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NEW SOUTH WALES

Higher School Certificate

Mathematics Extension 2

Exercise 4/67

by James Coroneos*

1. The complex numbers form a field. The elements of a field $\mathbb{F}$ obey the following axioms, (where a, b, c are elements of $\mathbb{F}$).

L1 $a + b \in \mathbb{F}$ *Closure law of addition*
L2 $a \times b \in \mathbb{F}$ *Closure law of multiplication*
L3 $a + b = b + a$ *Commutative law of addition*
L4 $a \times b = b \times a$ *Commutative law of multiplication*
L5 $(a + b) + c = a + (b + c)$ *Associative law of addition*
L6 $(a \times b) \times c = a \times (b \times c)$ *Associative law of multiplication*
L7 $a \times (b+c) = a \times b + a \times c$ *Distributive law of multiplication over addition*
L8 If $0 \in \mathbb{F}$ and $a + 0 = 0 + a = a$, then 0 is the *additive identity* of $\mathbb{F}$
L9 if $1 \in \mathbb{F}$ and $a \times 1 = 1 \times a = a$, then 1 is the *multiplicative identity* of $\mathbb{F}$.
L10 If $(-a)$ and 0 are elements of $\mathbb{F}$, and $a + (-a) = (-a) + a = 0$, then $(-a)$ is the *additive inverse* of a.
L11 If a^{-1} and 1 are elements of $\mathbb{F}$, and $a \times a^{-1} = a^{-1} \times = 1$, where $a \neq 0$, then a^{-1} is the *multiplicative inverse* of a.

If $z_1 = a + ib, z_2 = c + id, z_3 = e + if$ where a, b, c, d, e, f are real numbers, prove that the laws **L1** to **L11** above are satisfied by the complex numbers z_1, z_2, z_3. [Note that the additive and multiplicative identities are $0+0i, 1+0i$ respectively; and that the additive and multiplicative inverses of z_1 are $-z_1 = -a - ib, z_1^{-1} = \frac{a}{a^2+b^2} + i.\frac{-b}{a^2+b^2}$ respectively. z_1^{-1} only exists if $z_1 \neq 0$, i.e., if $a + ib \neq 0$, i.e., if $a \neq 0$ and $b \neq 0$]

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*Other resources by James Coroneos are available. Write to P.O. Box 25, Rose Bay, NSW, 2029, Australia, for a catalogue. TYPESET BY $\mathcal{AMS}$-TEX.

2. Which of the field properties **L1** to **L11** from question **1** hold for complex numbers of the form **(i)** $x + iy$ **(ii)** iy where $x, y \in$ **(a)** $\mathbb{N}$ **(b)** $\mathbb{Z}$ **(c)** $\mathbb{Q}$?

3. The definition of a group G under an operation $\odot$ is given below:

"If $a, b, c \in G$, then

(i) $a \odot b \in G$

(ii) $(a \odot b) \odot c = a \odot (b \odot c)$

(iii) $a \odot I = a$, where $I \in G$

(iv) $a \odot a^{-1} = I$, where a^{-1} is the 'inverse of a' with respect to $\odot$

If further, $a \odot b = b \odot a$, the group is called 'Abelian'."

Investigate whether the numbers $1, -1, i, -i$ form a group under the operation **(a)** $+$ **(b)** $\times$