Independent Trial HSC 2008 Mathematics Extension 1 Marking Guidelines

Question 1

a. Outcomes assessed : H5

Marking Guidelines

Criteria	Marks
• writes primitive and substitutes for x	1
• evaluates in simplest surd form	1

Answer

$$\int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \sec x \tan x \, dx = \Big[\sec x\Big]_{\frac{\pi}{4}}^{\frac{\pi}{3}} = \sec\tfrac{\pi}{3} - \sec\tfrac{\pi}{4} = 2 - \sqrt{2}$$

b. Outcomes assessed : PE3

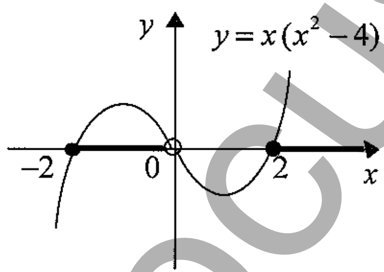
Marking Guidelines

Criteria	Marks
• writes an equivalent inequality not involving a variable denominator	1
• writes one inequality for x	1
• combines this with a second inequality for x	1

Answer

$$\frac{x^2-4}{x} \geq 0$$

$$x(x^2-4) \geq 0 \ , \quad x \neq 0$$



By inspection of the graph,
$-2 \leq x < 0$ or $x \geq 2$

c. Outcomes assessed : H5

Marking Guidelines

Criteria	Marks
i • finds gradient of tangent to $y = x^3$ at P	1
• finds gradient of tangent to $y = 1 - \ln x$ at P	1
ii • finds the acute angle between the lines correct to the nearest degree	1

Answer

i.

$$y = x^3$$
$$\frac{dy}{dx} = 3x^2$$
$$x = 1 \Rightarrow \frac{dy}{dx} = 3$$

Tangent at $P(1,1)$ has gradient 3

$$y = 1 - \ln x$$
$$\frac{dy}{dx} = -\frac{1}{x}$$
$$x = 1 \Rightarrow \frac{dy}{dx} = -1$$

Tangent at $P(1,1)$ has gradient -1

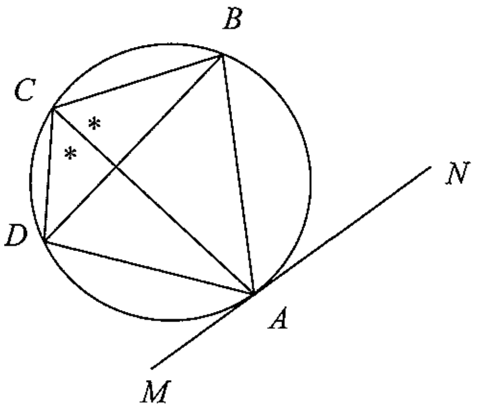
ii. $\tan\theta = \left|\dfrac{3-(-1)}{1+3\times(-1)}\right| = 2 \quad \Rightarrow \quad \theta \approx 63^\circ$ (to the nearest degree)

d. Outcomes assessed : PE2, PE3

Marking Guidelines

Criteria	Marks
• explains why angles *BAN*, *BCA* are equal	1
• explains why angles *DCA*, *DBA* are equal	1
• uses the equality of angles *BCA*, *DCA* to complete proof that angles *BAN*, *DBA* are equal	1
• quotes test for parallel lines to deduce tangent *MAN* is parallel to *BD*	1

Answer



$\angle BAN = \angle BCA$ (*angle between tangent and chord drawn to point of contact is equal to angle subtended by the chord in the alternate segment*)

$\angle BCA = \angle DCA$ (*given that AC bisects* $\angle BCD$)

$\angle DCA = \angle DBA$ (*angles subtended at the circumference by the same arc DA are equal*)

$\therefore\ MAN \parallel DB$ (*equal alternate angles on transversal BA since* $\angle BAN = \angle DBA$)

Question 2

a. Outcomes assessed : H3, H5

Marking Guidelines

Criteria	Marks
• writes condition on common ratio $\ln x$ for existence of limiting sum	1
• solves this inequality for x	1

Answer

Limiting sum of geometric series $1 + \ln x + (\ln x)^2 + \ldots$ exists for $-1 < \ln x < 1$

$\therefore$ since $f(x) = e^x$ is an increasing function, $e^{-1} < e^{\ln x} < e^1$

$\therefore\ \frac{1}{e} < x < e$

b. Outcomes assessed : H5

Marking Guidelines

Criteria	Marks
• finds x coordinate of P	1
• finds y coordinate of P as the sum of two surds	1
• simplifies surd expression for y	1

Answer

$A\left(8, \sqrt{8}\right)$ $B\left(50, \sqrt{50}\right)$

$2 : 1$

$$\left(\frac{100+8}{2+1}, \frac{2\sqrt{50}+\sqrt{8}}{2+1}\right)$$

$$\therefore\ P\left(36, \frac{10\sqrt{2}+2\sqrt{2}}{3}\right)$$

$$P\left(36, 4\sqrt{2}\right)$$

c. Outcomes assessed : H5

Marking Guidelines

Criteria	Marks
i • finds value of R	1
• finds value of α	1
ii • solves equation for x	1

Answer

i. $\cos x - \sqrt{3}\sin x = 2\left(\frac{1}{2}\cos x - \frac{\sqrt{3}}{2}\sin x\right)$

$= 2\left(\cos\frac{\pi}{3}\cos x - \sin\frac{\pi}{3}\sin x\right)$

$= 2\cos\left(x + \frac{\pi}{3}\right)$

ii. $\cos x - \sqrt{3}\sin x = -2\,,\quad 0 \le x \le 2\pi$

$\cos\left(x + \frac{\pi}{3}\right) = -1\,,\quad \frac{\pi}{3} \le x + \frac{\pi}{3} \le 2\pi + \frac{\pi}{3}$

$x + \frac{\pi}{3} = \pi$

$x = \frac{2\pi}{3}$

d. Outcomes assessed : PE3, PE4

Marking Guidelines

Criteria	Marks
i • shows by differentiation that tangent has gradient t	1
• finds the equation of the tangent	1
ii • substitutes coordinates of P to write equation for t	1
• solves equation for t	1

Answer

i. $y = t^2 \;\Rightarrow\; \dfrac{dy}{dt} = 2t$

$x = 2t \;\Rightarrow\; \dfrac{dx}{dt} = 2$

$\therefore \dfrac{dy}{dx} = \dfrac{dy}{dt} \div \dfrac{dx}{dt} = t$

Tangent at $T\left(2t\,,t^2\right)$ has gradient t

and equation $y - t^2 = t(x - 2t)$

$y - t^2 = tx - 2t^2$

$tx - y - t^2 = 0$

ii. $P\left(1\,,-2\right)$ lies on this tangent if

$t + 2 - t^2 = 0$

$t^2 - t - 2 = 0$

$(t-2)(t+1) = 0$

$t = 2$ or $t = -1$

Question 3

a. Outcomes assessed : PE5, HE4

Marking Guidelines

Criteria	Marks
• applies the product rule, obtaining first term	1
• obtains second term by deriving inverse sine	1

Answer

$$\frac{d}{dx}\left(x\sin^{-1}x\right) = \sin^{-1}x \;+\; \frac{x}{\sqrt{1-x^2}}$$

b. Outcomes assessed : H5, HE4

Marking Guidelines

Criteria	Marks
i • shows $f(x)$ is increasing for $x>1$	1
• shows the curve $y=f(x)$ is concave up for $x>1$	1
ii • sketches $y=f(x)$ showing endpoint and asymptote $y=x$	1
• sketches $y=f^{-1}(x)$ showing endpoint and asymptote	1
iii • makes x the subject	1
• interchanges x and y to find equation of inverse function	1

Answer

i. $f(x)=x+\dfrac{1}{x}$ for $x\geq 1$

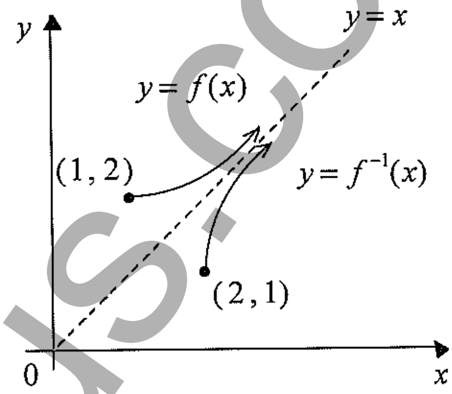
$f'(x)=1-\dfrac{1}{x^2}>0$ for $x>1$

$\therefore f(x)$ is increasing for $x>1$

$f''(x)=\dfrac{2}{x^3}>0$ for $x>1$

$\therefore y=f(x)$ is concave up for $x>1$

ii.



iii. $y=x+\dfrac{1}{x}$, $x\geq 1$ and $y\geq 2$

Rearrangement gives $x^2-xy+1=0$, $x\geq 1$ and $y\geq 2$

Considering this quadratic in x: $x=\dfrac{y\pm\sqrt{y^2-4}}{2}$, $x\geq 1$ and $y\geq 2$

Clearly the branch $x=\dfrac{y-\sqrt{y^2-4}}{2}$ contains points for which $x<1$.

Hence expressing x as the subject of $y=f(x)$, $x=\dfrac{y+\sqrt{y^2-4}}{2}$, $y\geq 2$.

Interchanging x and y, the inverse function is $f^{-1}(x)=\dfrac{x+\sqrt{x^2-4}}{2}$, $x\geq 2$

c. Outcomes assessed : HE2

Marking Guidelines

Criteria	Marks
• verifies truth of statement for $n=1$	1
• expresses LHS of $S(k+1)$ in terms of LHS of $S(k)$	1
• expresses LHS of $S(k+1)$ in terms of RHS of $S(k)$, conditional on truth of $S(k)$	1
• completes algebraic rearrangement to show $S(k+1)$ is true if $S(k)$ is true	1

Answer

Define the sequence of statements $S(n)$, $n=1,2,3,\ldots$ by $S(n):\ \dfrac{1}{2!}+\dfrac{2}{3!}+\dfrac{3}{4!}+\ldots+\dfrac{n}{(n+1)!}=1-\dfrac{1}{(n+1)!}$

Consider $S(1)$: $LHS=\frac{1}{2!}=\frac{1}{2}$ $RHS=1-\frac{1}{2!}=1-\frac{1}{2}=\frac{1}{2}$ $\therefore S(1)$ is true

If $S(k)$ is true: $\frac{1}{2!}+\frac{2}{3!}+\frac{3}{4!}+...+\frac{k}{(k+1)!}=1-\frac{1}{(k+1)!}$ $**$

Consider $S(k+1)$:
$$LHS=\left\{\frac{1}{2!}+\frac{2}{3!}+\frac{3}{4!}+...+\frac{k}{(k+1)!}\right\}+\frac{k+1}{(k+2)!}$$
$$=\left\{1-\frac{1}{(k+1)!}\right\}+\frac{k+1}{(k+2)!} \qquad \text{if } S(k) \text{ is true, using } **$$
$$=1-\frac{k+2}{(k+2)(k+1)!}+\frac{k+1}{(k+2)!}$$
$$=1-\frac{k+2-(k+1)}{(k+2)!}$$
$$=1-\frac{1}{(k+2)!}$$
$$=RHS$$

Hence if $S(k)$ is true, then $S(k+1)$ is true. But $S(1)$ is true, hence $S(2)$ is true, and then $S(3)$ is true and so on. Hence by Mathematical Induction, $S(n)$ is true for all positive integers $n\geq 1$.

Question 4

a. Outcomes assessed : H8

Marking Guidelines

Criteria	Marks
• expresses integrand in terms of $\cos 8x$	1
• finds primitive function	1

Answer

$$\int \cos^2 4x\, dx=\int \tfrac{1}{2}\left(1+\cos 8x\right) dx=\tfrac{1}{2}x+\tfrac{1}{16}\sin 8x+c$$

b. Outcomes assessed : H5, PE3

Marking Guidelines

Criteria	Marks
i • uses cosine rule and trigonometric identity to find AB in terms of $\sin\frac{1}{2}\theta$	1
• adds arc length to AB, equating sum and diameter to obtain required equation	1
ii • shows $f(1)$ and $f(2)$ have opposite signs	1
• uses continuity of $f(\theta)$ to deduce equation has root between 1 and 2.	1
iii • applies Newton's rule, substituting $\theta=1$	1
• evaluates expression to obtain next approximation, giving result correct to 1 dec. place	1

Answer

i. Using cosine rule, $AB^2 = 1^2 + 1^2 - 2\cos\theta$

$$AB^2 = 2(1-\cos\theta)$$
$$= 4\sin^2\tfrac{1}{2}\theta$$
$$\therefore AB = 2\sin\tfrac{1}{2}\theta$$

$\therefore$ Perimeter = diameter $\Rightarrow \theta + 2\sin\tfrac{1}{2}\theta = 2$

$$\theta + 2\sin\tfrac{1}{2}\theta - 2 = 0$$

ii. Let $f(\theta) = \theta + 2\sin\tfrac{1}{2}\theta - 2$

$$f(1) = -1 + 2\sin\tfrac{1}{2} \approx -0\cdot04 < 0$$
$$f(2) = 2\sin 1 \approx 1\cdot68 > 0$$

Since $f(\theta)$ is continuous,
$f(\theta) = 0$ for some $1 < \theta < 2$.

iii. $f(\theta) = \theta + 2\sin\tfrac{1}{2}\theta - 2$

$f'(\theta) = 1 + \cos\tfrac{1}{2}\theta$

$$\theta_1 = \theta_0 - \frac{f(\theta_0)}{f'(\theta_0)}$$

$$\theta_1 = 1 - \frac{-1 + 2\sin\frac{1}{2}}{1 + \cos\frac{1}{2}}$$

$\therefore \theta_1 \approx 1\cdot0$ (to 1 dec. place)

c. Outcomes assessed : HE6

Marking Guidelines

Criteria	Marks
• writes dx in terms of du	1
• writes integrand in terms of u and changes limits to u values	1
• finds primitive function	1
• evaluates in simplest exact form	1

Answer

$x = u^2,\ u \ge 0$

$dx = 2u\,du$

$x = 1 \Rightarrow u = 1$

$x = 25 \Rightarrow u = 5$

$$\int_1^{25} \frac{1}{x + \sqrt{x}}\,dx = \int_1^5 \frac{1}{u(u+1)}\,.\,2u\,du$$
$$= 2\Big[\ln(u+1)\Big]_1^5$$
$$= 2\left(\ln 6 - \ln 2\right)$$
$$= 2\ln 3$$

Question 5

a. Outcomes assessed : PE3

Marking Guidelines

Criteria	Marks
i • shows $P(1) = 0$	1
ii • uses product of roots is 1 to deduce 3rd root is reciprocal of α	1
iii • writes sum of squares of roots in terms of square of sum and sum of two-way products	1
• uses relationships between coefficients of polynomial equation and its roots	1

Answer

i. $P(x) = x^3 - kx^2 + kx - 1$

$P(1) = 1 - k + k - 1 = 0$

ii. Product of the roots is 1.

Hence if the roots are $1, \alpha, \beta$,

then $\alpha\beta = 1$. $\therefore \frac{1}{\alpha}$ is the 3rd root.

iii. $\alpha^2 + \beta^2 + \gamma^2 = (\alpha + \beta + \gamma)^2 - 2(\alpha\beta + \beta\gamma + \gamma\alpha)$

$$\therefore \alpha^2 + \tfrac{1}{\alpha^2} + 1^2 = k^2 - 2k$$
$$\therefore\ \alpha^2 + \tfrac{1}{\alpha^2} = k^2 - 2k - 1$$

b. Outcomes assessed : HE3

Marking Guidelines

Criteria	Marks
i • counts the number of codes with all three digits different	1
• divides by the total number of codes to find the probability	1
ii • counts the number of codes with exactly two digits the same	1
• writes the probability of such a code	1

Answer

i. $P(all\ \ different) = \dfrac{9\times 8\times 7}{9\times 9\times 9} = \dfrac{56}{81}$

ii. Consider code of form A, A, B or A, B, A or B, A, A
Number of such codes is $9\times 8\times 3$

$$P(exactly\ two\ the\ same) = \frac{9\times 8\times 3}{9\times 9\times 9} = \frac{8}{27}$$

c. Outcomes assessed : HE4, HE5

Marking Guidelines

Criteria	Marks
i • finds θ in terms of x	1
ii • derives θ with respect to x	1
• finds the derivative of θ with respect to t in terms of x	1
• states the rate at which θ is changing when $x = 20$	1

Answer

i. $\tan\theta = \frac{40}{x}$, $0 < \theta < \frac{\pi}{2}$

$\therefore\ \theta = \tan^{-1}\frac{40}{x}$

ii.
$$\frac{d\theta}{dt} = \frac{d\theta}{dx}\cdot\frac{dx}{dt}$$
$$= \frac{1}{1+\frac{1600}{x^2}}\cdot\frac{-40}{x^2}\ .5$$
$$= \frac{-200}{x^2+1600}$$
$$\therefore x = 20\ \Rightarrow \frac{d\theta}{dt} = -\frac{1}{10}$$

θ is decreasing at a rate of $0\cdot 1$ radians per second.

Question 6

a. Outcomes assessed : H3, HE3, HE5

Marking Guidelines

Criteria	Marks
i • finds a in terms of x	1
ii • finds t as a function of x by integration	1
• rearranges to find x as a function of t	1
iii • finds t when $x = 0$	1
iv • shows intercepts on the axes	1
• shows asymptote $x = 2$	1

Answer

i. $v = 2 - x$

$a = v\dfrac{dv}{dx}$

$= (2-x).(-1)$

$= x - 2$

ii. $\dfrac{dx}{dt} = 2 - x$

$\dfrac{dt}{dx} = \dfrac{1}{2-x}$

$t = -\ln A(2-x), \quad A$ constant

$\left.\begin{array}{l} t = 0 \\ x = -4 \end{array}\right\} \Rightarrow \begin{array}{l} 6A = 1 \\ A = \frac{1}{6} \end{array}$

$\therefore\ -t = \ln\left(\dfrac{2-x}{6}\right)$

$e^{-t} = \dfrac{2-x}{6}$

$6\,e^{-t} = 2 - x$

$\therefore\quad x = 2 - 6e^{-t}$

iii. When $t = 0,\ x = -4\ \ \therefore v > 0$

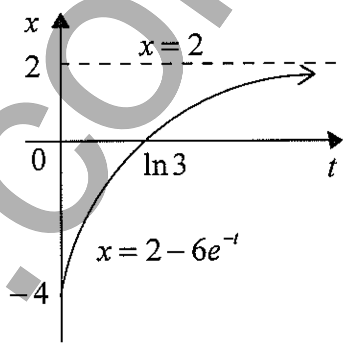
Particle is initially moving right, and it continues moving right approaching $x = 2$.

Hence particle has travelled 4 metres from its starting point when $x = 0$.

$x = 0 \Rightarrow -t = \ln\frac{1}{3}$.

$\therefore$ particle travels first 4 metres in $\ln 3$ seconds.

iv.



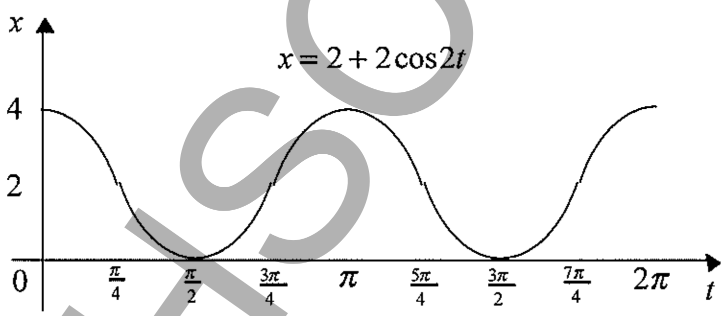
b. Outcomes assessed : HE3

Marking Guidelines

Criteria	Marks
i • sketches curve with correct shape and position showing intercept on x axis	1
• shows intercepts on t axis for at least one period	1
ii • differentiates to find $\ddot{x}$ as a function of t, and hence as a function of x	1
iii • states the period of the motion	1
iv • finds x when $t = 2$	1
• states the distance travelled in the first 2 seonds	1

Answer

i.



ii. $\dot{x} = -4\sin 2t$

$\ddot{x} = -4\left(2\cos 2t\right)$

$= -4(x-2)$

iii. Period is π seconds

iv. $t = 2 \Rightarrow x = 2 + 2\cos 4 \approx 0\cdot 69$

But $\frac{\pi}{2} < 2 < \frac{3\pi}{4}$. Hence by inspection of the graph, particle has travelled $4\cdot 7$ m (correct to 2 sig. fig.)

Question 7

a. Outcomes assessed : HE3

Marking Guidelines

Criteria	Marks
i • writes expressions for x and y	1
• finds x when $y=0$ and hence required expression for R	1
ii • calculates R for $V=20$ when $\theta=15°$, $\theta=45°$	1
• identifies region that can be watered	1
• finds the area of at least part of this region	1
• finds the total area in simplest exact form	1

Answer

i. $x = Vt\cos\theta \qquad y = -\frac{1}{2}gt^2 + Vt\sin\theta$

$x = R$ when $y = 0$ and $t \neq 0$

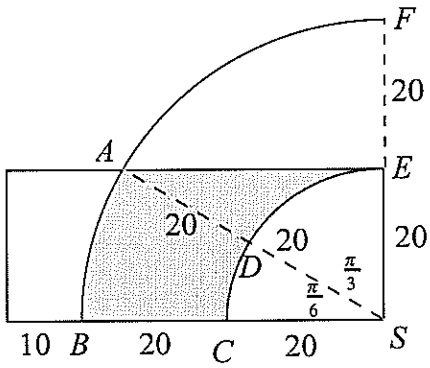
$y=0,\ t\neq 0 \ \Rightarrow\ V\sin\theta = \frac{1}{2}gt$

$$t = \frac{2V\sin\theta}{g}$$

$$\therefore R = \frac{V^2(2\sin\theta\cos\theta)}{g} = \frac{V^2\sin 2\theta}{g}$$

ii. $V=20,\ \theta=15° \ \Rightarrow\ R=20$

$V=20,\ \theta=45° \ \Rightarrow\ R=40$



The area of lawn that can be watered is shaded on the diagram.

Since $\cos\angle ESA = \frac{20}{40}$, $\angle ESA = \frac{\pi}{3}$ and $\angle ASB = \frac{\pi}{6}$.

$$Area = Sector\ ABS + \Delta AES - Quadrant\ ECS$$

$$= \tfrac{1}{2}\times 40^2 \times \tfrac{\pi}{6} + \tfrac{1}{2}\times 40\times 20\sin\tfrac{\pi}{3} - \tfrac{1}{4}\times\pi\times 20^2$$

$$= 100\times\tfrac{\pi}{3} + 200\sqrt{3}$$

Area is $100\left(\frac{\pi}{3} + 2\sqrt{3}\right)$ square metres.

b. Outcomes assessed : HE3

Marking Guidelines

Criteria	Marks
i • writes binomial expansion	1
ii • substitutes $x=1$ to deduce required result	1
iii • finds primitive of LHS of i.	1
• finds primitive of RHS of i.	1
• evaluates definite integrals of LHS and RHS between limits 0 and 1	1
• uses result from ii. to deduce required result	1

Answer

i.

$(1+x)^n \equiv {}^nC_0 + {}^nC_1 x + {}^nC_2 x^2 + \ldots + {}^nC_n x^n$

ii.

$x=1 \Rightarrow\ 2^n = {}^nC_0 + {}^nC_1 + {}^nC_2 + \ldots + {}^nC_n$

But ${}^nC_0 = 1 \qquad \therefore \sum_{r=1}^{n} {}^nC_r = 2^n - 1$

iii.

$$\left[\tfrac{1}{n+1}(1+x)^{n+1}\right]_0^1 = \left[{}^nC_0 x + {}^nC_1 \tfrac{1}{2}x^2 + \ldots + {}^nC_n \tfrac{1}{n+1}x^{n+1}\right]_0^1$$

$$\tfrac{1}{n+1}\left(2^{n+1}-1\right) = {}^nC_0 + \tfrac{1}{2}{}^nC_1 + \tfrac{1}{3}{}^nC_2 + \ldots + \tfrac{1}{n+1}{}^nC_n$$

$$\therefore \frac{1}{n+1}\sum_{r=1}^{n+1} {}^{n+1}C_r = \sum_{r=0}^{n} \frac{{}^nC_r}{r+1} \qquad \text{(using ii. with } n\to n+1\text{)}$$

Independent Trial HSC Examination 2008 Mathematics Extension 1 Mapping Grid

Question	Marks	Content	Syllabus Outcomes	Targeted Performance Bands
1 a	2	Trigonometric functions	H5	E2-E3
b	3	Inequalities	PE3	E2-E3
c i	2	Gradient of a tangent to a curve; Logarithmic functions	H5	E2-E3
ii	1	Angle between two lines	H5	E2-E3
d	4	Circle geometry	PE2, PE3	E2-E3
2 a	2	Series; Exponential and logarithmic functions	H3, H5	E2-E3
b	3	Division of an interval	H5	E2-E3
c i	2	Further trigonometry	H5	E2-E3
ii	1	Further trigonometry	H5	E2-E3
d i	2	Parametric representation	PE4	E2-E3
ii	2	Parametric representation	PE3	E2-E3
3 a	2	Rules of differentiation; Inverse trigonometric functions	PE5, HE4	E2-E3
b i	2	Geometrical applications of differentiation	H5	E2-E3
ii	2	Inverse functions	HE4	E2-E3
iii	2	Inverse functions	HE4	E2-E3
c	4	Mathematical induction	HE2	E3-E4
4 a	2	Integration	H8	E2-E3
b i	2	Trigonometric functions	H5	E2-E3
ii	2	Polynomials	PE3	E2-E3
iii	2	Iterative methods	PE3	E2-E3
c	4	Methods of integration	HE6	E2-E3
5 a i	1	Polynomials	PE3	E2-E3
ii	1	Polynomials	PE3	E2-E3
iii	2	Polynomials	PE3	E2-E3
b i	2	Further probability	HE3	E2-E3
ii	2	Further probability	HE3	E2-E3
c i	1	Inverse trigonometric functions	HE4	E2-E3
ii	3	Rates of change	HE5	E2-E3
6 a i	1	Velocity, acceleration as functions of x	HE5	E3-E4
ii	2	Velocity, acceleration as functions of x	HE5	E3-E4
iii	1	Exponential and logarithmic functions	H3	E3-E4
iv	2	Exponential growth and decay	HE3	E3-E4
b i	2	Simple harmonic motion	HE3	E3-E4
ii	1	Simple harmonic motion	HE3	E3-E4
iii	1	Simple harmonic motion	HE3	E3-E4
iv	2	Simple harmonic motion	HE3	E3-E4
7 a i	2	Projectile motion	HE3	E3-E4
ii	4	Projectile motion	HE3	E3-E4
b i	1	Binomial expansion	HE3	E3-E4
ii	1	Binomial expansion	HE3	E3-E4
iii	4	Binomial expansion	HE3	E3-E4