Student Number ....

Barker College

2011
TRIAL
HIGHER SCHOOL CERTIFICATE

Mathematics Extension 1

Staff Involved: **AM FRIDAY 12 AUGUST**

- **PJR***
- **MRB**
- **KJL**
- **GPF**
- **GIC***
- **GDH**
- **RMH**

105 copies

General Instructions

- Reading time – 5 minutes
- Working time – 2 hours
- Write using blue or black pen
- Make sure your Barker Student Number is on ALL pages of your solutions
- Board-approved calculators may be used
- A table of standard integrals is provided at the back of this paper
- ALL necessary working should be shown in every question

Total marks – 84

- Attempt Questions 1 – 7
- All questions are of equal value
- Marks may be deducted for careless or poorly arranged working

Total marks – 84
Attempt Questions 1–7
ALL questions are of equal value

Answer each question on a SEPARATE sheet of paper

Marks

Question 1 (12 marks) **[START A NEW PAGE]**

(a) The point $P(x,\ y)$ divides the interval AB internally in the ratio $2:1$ **2**

If A is the point (6, 1) and B is the point (12, -8), find the coordinates of $P(x,\ y)$

(b) Evaluate $\lim_{x \to 0}\left(\frac{\tan x}{3x}\right)$ **2**

(c) Use the table of standard integrals to evaluate $\int_0^{\frac{\pi}{2}} \sec\frac{x}{2}\tan\frac{x}{2}\,dx$ **2**

(d) Solve $\frac{x}{x-4} \leq 2$ **3**

(e) Evaluate $\int_{-\frac{1}{4}}^{\frac{1}{4}} \frac{dx}{\sqrt{1-4x^2}}$ **3**

Marks

Question 2 (12 marks) **[START A NEW PAGE]**

(a) Find the acute angle between the curves $y = \log_e x$ and $y = 1 - x^2$ at the point P (1, 0) **3**

Give your answer correct to the nearest minute.

(b) The point $P\left(2ap, ap^2\right)$ is a point on the parabola $x^2 = 4ay$ with focus $S\left(0, a\right)$

(i) Find M, the midpoint of the chord OP, where O is the origin **1**

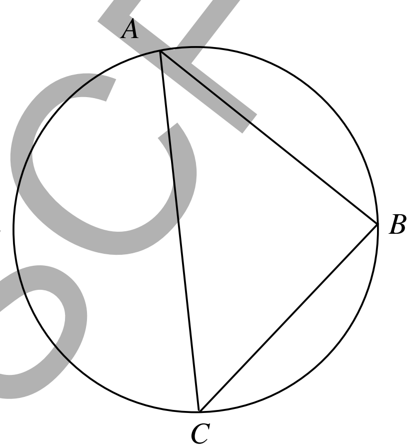
(ii) Find the gradient of the chord OP **1**

(iii) Find the point A on the parabola where the tangent is parallel to the chord OP **2**

(iv) Show that A is equidistant from M and the x-axis **1**

(c) ΔABC is inscribed in a circle as shown below.

The tangent at C meets AB produced at P and the bisector of $\angle ACB$ meets AB at Q



(i) Copy and complete the diagram **1**

(ii) Prove that $PC = PQ$ **3**

Marks

Question 3 (12 marks) **[START A NEW PAGE]**

(a) Let $f(x) = \ln(\tan x)$, where $0 < x < \frac{\pi}{2}$ **3**

Show that $f'(x) = 2\operatorname{cosec} 2x$

(b) Use the substitution $x = 2\sin\theta$ to evaluate $\int_0^1 \sqrt{4 - x^2}\, dx$ **3**

(c) (i) State the domain and range of the function $f(x) = \cos^{-1} 2x$ **2**

(ii) Draw a neat sketch of the function $f(x) = \cos^{-1} 2x$ **1**

Clearly label all essential features

(iii) Find the equation of the tangent to the curve $f(x) = \cos^{-1} 2x$ at the point where the curve crosses the y-axis. **3**

Marks

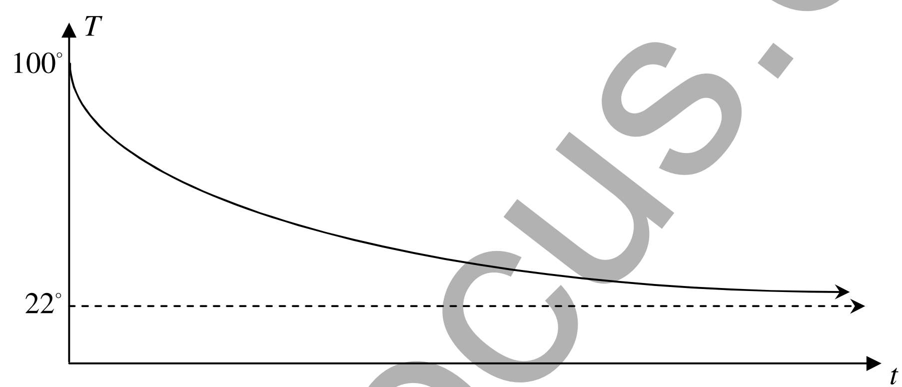
Question 4 (12 marks) **[START A NEW PAGE]**

(a) (i) Show that $\sin(A + B) + \sin(A - B) = 2\sin A \cos B$ **1**

(ii) Hence, or otherwise, evaluate $\int_0^{\pi/4} \sin 4x \cos 2x \, dx$ **3**

(b) If $f(x+2) = x^2 + 2$, find $f(x)$ **2**

(c) The graph shown below represents the relationship between T, the temperature in C° of a cooling cup of coffee, and t, the time in minutes.



The rate of cooling of this coffee is given by $\frac{dT}{dt} = -k(T - A)$, where k and A are constants and $k > 0$.

(i) Show that $T = A + Be^{-kt}$ is a solution to the differential equation $\frac{dT}{dt} = -k(T - A)$, given that B is a constant. **1**

(ii) By examining the graph when $t = 0$ and $t \to \infty$, find the values of A and B **2**

(iii) If the temperature of the coffee is $50^\circ C$ after 90 minutes, show that $k = -\frac{1}{90}\ln\left(\frac{14}{39}\right)$ **2**

(iv) Hence, find the rate at which the coffee is cooling after 90 minutes. Give your answer correct to two significant figures. **1**

Marks

Question 5 (12 marks) **[START A NEW PAGE]**

(a) Evaluate $\int_0^{\frac{\pi}{4}} \cos x \sin^2 x \, dx$ **2**

(b) The volume of a sphere is increasing at the rate of $5\,cm^3$ per second. **3**

At what rate is the surface area increasing when the radius is $20\,cm$?

(c) A particle moves in such a way that its displacement x cm from an origin O at any time t seconds is given by the function $x = 4 + \sqrt{3}\cos 3t - \sin 3t$

(i) Show that the particle is moving in simple harmonic motion. **2**

(ii) Express $\sqrt{3}\cos 3t - \sin 3t$ in the form $R\cos(3t + \alpha)$, where α is acute and in radians. **2**

(iii) Find the amplitude of the motion. **1**

(iv) Find when the particle first passes through the centre of motion. **2**

Marks

Question 6 (12 marks) **[START A NEW PAGE]**

(a) Show by induction that $7^n + 2$ is divisible by 3, for all positive integers n **3**

(b) Given the function $f(x) = \dfrac{2x+1}{x-1}$

(i) Find any vertical and horizontal asymptotes **1**

(ii) State the domain of the inverse function $f^{-1}(x)$ **1**

(iii) Sketch the graph of the inverse function $f^{-1}(x)$ **2**
Clearly label all critical features of the inverse function $f^{-1}(x)$

(c) A particle is moving along the x-axis so that its acceleration after t seconds is given by

$$\ddot{x} = -e^{-\frac{x}{2}}$$

The particle starts at the origin with an initial velocity of $2\,cm/\sec$

(i) If v is the velocity of the particle, find v^2 as a function of x **2**

(ii) Show that the displacement x as a function of time t is given by **3**

$$x = 4\log_e\left(\frac{t+2}{2}\right)$$

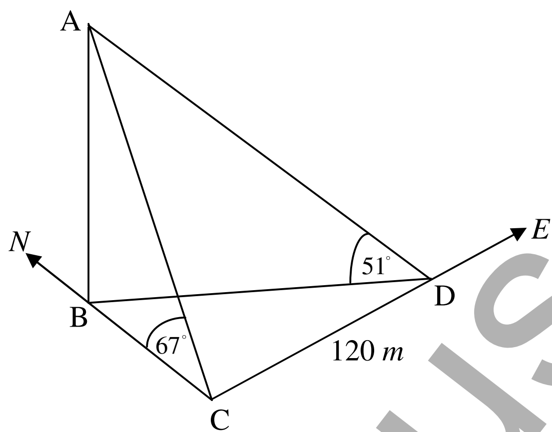
Marks

Question 7 (12 marks) **[START A NEW PAGE]**

(a) James is standing at the top A of a tower AB which is built on level ground.

From point C, due south of the base B of the tower, the angle of elevation of the top A of the tower is 67°

From point D, which is $120\,m$ due east of point C, the angle of elevation of the top A of the tower is 51°



(i) Calculate the height of the tower AB (to the nearest metre) **3**

(ii) James projects a stone horizontally from the top of the tower with velocity $V\ m/s$

If this stone lands at point D, find the value of V **3**

(Give your answer correct to one decimal place)

You may assume the equations of motion are

$x = vt\cos\theta$ and $y = vt\sin\theta - 5t^2$ (Do **NOT** prove this)

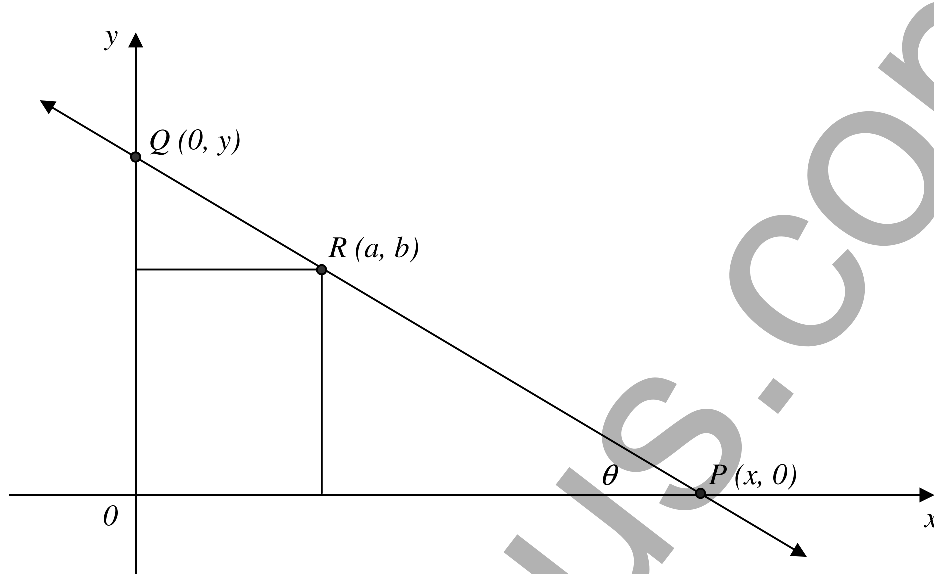
(Hint: Use point A as the origin)

Question 7 continues on page 9

Question 7 (continued)

(b) The point $R(a, b)$ lies in the positive quadrant of the number plane.

A line through R meets the positive x and y axes at P and Q respectively and makes an angle θ with the x-axis.



(i) Show that the length of PQ is equal to $\dfrac{a}{\cos\theta} + \dfrac{b}{\sin\theta}$ **2**

(ii) Hence, show that the minimum length of PQ is equal to $\left(a^{2/3} + b^{2/3}\right)^{3/2}$ **4**

End of Question 7

End of Paper

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STANDARD INTEGRALS

$$\int x^n\,dx = \frac{1}{n+1}x^{n+1}, \quad n \neq -1; \quad x \neq 0, \text{ if } n < 0$$

$$\int \frac{1}{x}\,dx = \ln x, \quad x > 0$$

$$\int e^{ax}\,dx = \frac{1}{a}e^{ax}, \quad a \neq 0$$

$$\int \cos ax\,dx = \frac{1}{a}\sin ax, \quad a \neq 0$$

$$\int \sin ax\,dx = -\frac{1}{a}\cos ax, \quad a \neq 0$$

$$\int \sec^2 ax\,dx = \frac{1}{a}\tan ax, \quad a \neq 0$$

$$\int \sec ax \tan ax\,dx = \frac{1}{a}\sec ax, \quad a \neq 0$$

$$\int \frac{1}{a^2+x^2}\,dx = \frac{1}{a}\tan^{-1}\frac{x}{a}, \quad a \neq 0$$

$$\int \frac{1}{\sqrt{a^2-x^2}}\,dx = \sin^{-1}\frac{x}{a}, \quad a > 0, \quad -a < x < a$$

$$\int \frac{1}{\sqrt{x^2-a^2}}\,dx = \ln\left(x+\sqrt{x^2-a^2}\right), \quad x > a > 0$$

$$\int \frac{1}{\sqrt{x^2+a^2}}\,dx = \ln\left(x+\sqrt{x^2+a^2}\right)$$

NOTE: $\ln x = \log_e x, \quad x > 0$

YEAR 12 MATHEMATICS EXTENSION 1 TRIAL HSC

20 August 2011

Question 1

a) $m:n = 2:1$

$P = \left(\frac{mx_2+nx_1}{m+n}, \frac{my_2+ny_1}{m+n}\right)$

$= \left(\frac{2\times12+1\times6}{3}, \frac{2\times-8+1\times1}{3}\right)$

$= \left(\frac{24+6}{3}, \frac{-16+1}{3}\right)$

$= (10, -5)$

$\frac{1}{3}\lim_{x\to0}\frac{\tan x}{x} = \frac{1}{3}\times1 = \frac{1}{3}$

$\left[2\sec\frac{x}{2}\right]_0^{\frac{\pi}{2}}$

$2\left(\sec\frac{\pi}{4} - \sec 0\right)$

$2\times(\sqrt{2}-1)$

$= 2(\sqrt{2}-1)$

1st method: multiplying both sides by the square of the denominator

$(x-4)^2\times\frac{x}{x-4} \le 2(x-4)^2$

$(x-4)x \le 2(x^2-8x+16)$

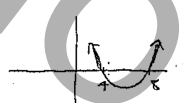
$x^2-4x \le 2x^2-16x+32$

$\therefore 0 \le x^2-12x+32$

$\therefore x^2-12x+32 \ge 0$

$(x-4)(x-8) \ge 0$

zeros are $x=4$, $x=8$



$x\le4$, $x\ge8$

but $x\ne4$ because it is a zero of the denominator

$\therefore x<4$ or $x\ge8$

2nd Method: graphical

for $\frac{x}{x-4}\le2$

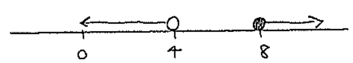
1. $x\ne4$

2. solve $\frac{x}{x-4} = 2$

$x = 2x-8$

$x=8$

$\therefore$ critical points are $x=4, 8$



check all 3 regions:

i) $x<4$; $x=0$, $\frac{0}{0-4}\le2$

true $\therefore x<4$

ii) $4<x\le8$, $x=5$ $\frac{5}{1}\le2$

not true

iii) $x\ge8$, $x=9$, $\frac{9}{5}\le2$

$\therefore x<4$ or $x\ge8$

e) $\int_{-\frac{1}{4}}^{\frac{1}{4}}\frac{dx}{\sqrt{1-(2x)^2}}$

$= \frac{1}{2}\int_{-\frac{1}{4}}^{\frac{1}{4}}\frac{2\,dx}{\sqrt{1-(2x)^2}}$

$= \frac{1}{2}\left[\sin^{-1}2x\right]_{-\frac{1}{4}}^{\frac{1}{4}}$

$= \frac{1}{2}\left(\sin^{-1}\frac{1}{2} - \sin^{-1}\left(-\frac{1}{2}\right)\right)$

$= \frac{1}{2}\left(\sin^{-1}\frac{1}{2} + \sin^{-1}\frac{1}{2}\right)$

$= \frac{1}{2}\times2\sin^{-1}\frac{1}{2}$

$= \frac{\pi}{6}$

Question 2

a) $\tan\theta = \left|\frac{m_1-m_2}{1+m_1m_2}\right|$

$y_1 = \log_e x$ $\quad y_2 = 1-x^2$

$y_1' = \frac{1}{x}$ $\quad y_2' = -2x$

at $x=1$, $m_1 = y_1' = \frac{1}{1} = 1$

$x=1$, $m_2 = y_2' = -2\times1 = -2$

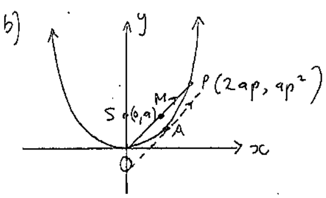
$\therefore \tan\theta = \left|\frac{1--2}{1+-2}\right|$

$= \left|\frac{3}{-1}\right|$

$\tan\theta = 3$

$\theta = \tan^{-1}3 = 71°34'$

b)



i) $M = \left(\frac{0+2ap}{2}, \frac{0+ap^2}{2}\right)$

$M = \left(ap, \frac{ap^2}{2}\right)$

ii) Gradient of OP $= \frac{ap^2-0}{2ap-0}$

$\therefore m_{OP} = \frac{p}{2}$

iii) A lies on the parabola

$x^2=4ay$ $\therefore y = \frac{x^2}{4a}$

$y' = \frac{2x}{4a} = \frac{x}{2a}$

now $y' = \frac{p}{2}$ since chords are parallel

$\frac{x}{2a} = \frac{p}{2}$

$x = ap$

if $x = ap$, $y = \frac{(ap)^2}{4a} = \frac{ap^2}{4}$

$\therefore A\left(ap, \frac{ap^2}{4}\right)$

iv) Distance of A from x-axis = its y value

$\therefore d_1 = \frac{ap^2}{4}$ units

istance from $A\left(ap, \frac{ap^2}{4}\right)$

$M\left(ap, \frac{ap^2}{2}\right)$ is

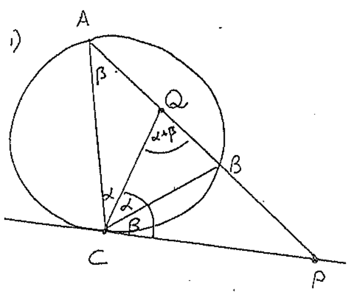
$\sqrt{(ap-ap)^2 + \left(\frac{ap^2}{2} - \frac{ap^2}{4}\right)^2}$

$\sqrt{0 + \left(\frac{ap^2}{4}\right)^2}$

$\frac{ap^2}{4}$ = distance d_1 from x-axis

A is equidistant from M and the x-axis.

i)



To prove that $PC = PQ$ it is easiest to prove that $\angle QCP = \angle CQP$

$\angle ACQ = \angle QCB = \alpha$ since CQ is the bisector

let $\angle BCP = \beta$

$\therefore \angle CAB = \beta$ (angle in the alternate segment equals angle between chord and tangent)

now $\angle CQP = \alpha + \beta$ (exterior angle of $\triangle AQC$)

$\angle QCP = \alpha + \beta$ (adjacent angles)

$\therefore \angle CQP = \angle QCP$

$\therefore \triangle PCQ$ is isosceles (base angles are equal)

$\therefore PC = PQ$ (equal sides in isosceles $\triangle$)

Question 3

a) $f'(x) = \frac{\sec^2 x}{\tan x}$ ← $f'(x)$, ← $f(x)$

LHS $= \frac{\frac{1}{\cos^2 x}}{\frac{\sin x}{\cos x}}$

$= \frac{1}{\cos^2 x} \times \frac{\cos x}{\sin x}$

$= \frac{1}{\cos x} \times \frac{1}{\sin x}$

$= \frac{2}{2\cos x \sin x}$

$= \frac{2}{\sin 2x}$

$= 2\,\mathrm{cosec}\, 2x$

$=$ RHS

b) $x = 2\sin\theta$

$dx = 2\cos\theta \, d\theta$

when $x = 0$, $\theta = 0$

$x = 1$, $\sin\theta = \frac{1}{2}$, $\theta = \frac{\pi}{6}$

$\therefore \int_0^{\frac{\pi}{6}} \sqrt{4 - (2\sin\theta)^2} \times 2\cos\theta \, d\theta$ ii)

$= \int_0^{\frac{\pi}{6}} \sqrt{4(1-\sin^2\theta)} \times 2\cos\theta \, d\theta$

$= \int_0^{\frac{\pi}{6}} \sqrt{4\cos^2\theta} \times 2\cos\theta \, d\theta$

$= \int_0^{\frac{\pi}{6}} 2\cos\theta \times 2\cos\theta \, d\theta$

$= 4\int_0^{\frac{\pi}{6}} \cos^2\theta \, d\theta$

now $\cos^2\theta = \frac{1+\cos 2\theta}{2}$

$\therefore = 4\int_0^{\frac{\pi}{6}} \frac{1+\cos 2\theta}{2} \, d\theta$

$= 2\int_0^{\frac{\pi}{6}} 1 + \cos 2\theta \, d\theta$

$= 2\left[\theta + \frac{\sin 2\theta}{2}\right]_0^{\frac{\pi}{6}}$

$= 2\left(\frac{\pi}{6} + \frac{\sin\frac{\pi}{3}}{2} - 0 - 0\right)$

$= 2\left(\frac{\pi}{6} + \frac{\sqrt{3}}{4}\right)$

$= \frac{\pi}{3} + \frac{\sqrt{3}}{2}$

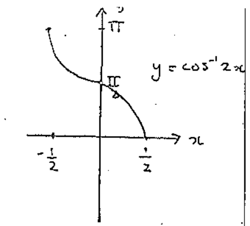
c) i) $f(x) = \cos^{-1} 2x$

Domain: $-1 \le 2x \le 1$

$-\frac{1}{2} \le x \le \frac{1}{2}$

Range: $0 \le y \le \pi$

ii)



iii) Gradient of tangent

$f'(x) = \frac{-2}{\sqrt{1-(2x)^2}}$

$= \frac{-2}{\sqrt{1-4x^2}}$

at $\left(0, \frac{\pi}{2}\right)$ $f'(0) = -\frac{2}{1} = -2 = m$

$\therefore$ Eqn of tangent

$y - \frac{\pi}{2} = -2(x-0)$

$y - \frac{\pi}{2} = -2x$

$2x + y - \frac{\pi}{2} = 0$

(or $y = -2x + \frac{\pi}{2}$)

Question 4

a) i) LHS $= \sin A\cos B + \cos A \sin B + \sin A \cos B - \cos A \sin B$

$= 2\sin A\cos B$

$=$ RHS

ii) $\frac{1}{2}\int_0^{\frac{\pi}{4}} 2\sin 4x \cos 2x \, dx$

$A = 4x \quad B = 2x$

$\therefore A + B = 6x \quad A - B = 2x$

$\frac{1}{2}\int_0^{\frac{\pi}{4}} \sin 6x + \sin 2x \, dx$

$\frac{1}{2}\left[\frac{-\cos 6x}{6} - \frac{\cos 2x}{2}\right]_0^{\frac{\pi}{4}}$

$= -\frac{1}{2}\left(\frac{\cos \frac{3\pi}{2}}{6} + \frac{\cos\frac{\pi}{2}}{2} - \left(\frac{\cos 0}{6} + \frac{\cos 0}{2}\right)\right)$

$-\frac{1}{2}\left(0 + 0 - \left(\frac{1}{6} + \frac{1}{2}\right)\right)$

$= -\frac{1}{2} \times -\frac{2}{3}$

$= \frac{1}{3}$

) 1st method :

$f(x+2) = (x+2)^2 - 4x - 2$

rewrite RHS in terms of $(x+2)^2$

$f(x+2) = (x+2)^2 - 4(x+2) + 6$

$f(x) = \underline{x^2 - 4x + 6}$

(replace $(x+2)$ with x)

2nd method :

$f(x) = f((x-2)+2)$

$= (x-2)^2 + 2$

$= x^2 - 4x + 4 + 2$

$= \underline{x^2 - 4x + 6}$

3rd method :

Equate $x^2 + 2 \equiv A(x+2)^2 + B(x+2) + C$

sub $x = -2$

$\therefore 6 \equiv 0 + 0 + C$

$\underline{c = 6}$

sub $x = 0$, $2 = 4A + 2B + c$

$2 = 4A + 2B + 6$

$\underline{2A + B = -2}$

sub $x = -1$, $3 = A + B + C$

$A + B = -3$

$\therefore$ solve simultaneously

$2A + B = -2$ ①

$A + B = -3$ ②

①−② $A = 1$

$\therefore B = -4$

$\therefore x^2 + 2 = (x+2)^2 - 4(x+2) + 6$

$\therefore f(x+2) = (x+2)^2 - 4(x+2) + 6$

$f(x) = \underline{x^2 - 4x + 6}$

c) i) Method 1

$T = A + Be^{-kt}$

$\frac{dT}{dt} = -kBe^{-kt}$

$= -k(A + Be^{-kt} - A)$

$= -k(T - A)$

$\therefore \frac{dT}{dt} = -k(T - A)$

Method 2

$T = A + Be^{-kt}$

show that $\frac{dT}{dt} = -k(T-A)$

$LHS = \frac{dT}{dt} = -Bke^{-kt}$

$RHS = -k(A + Be^{-kt} - A)$

$= -k(Be^{-kt})$

$= -Bke^{-kt}$

$= LHS$

ii) from graph $t = 0$, $T = 100°$

sub $t = 0$, $100 = A + Be^0$

$\underline{A + B = 100}$

$t \to \infty$, $T \to 22°$

as $t \to \infty$ $Be^{-kt} \to 0$

$22 = A + 0$

$\therefore \underline{A = 22}$

and $\underline{B = 78}$

so $T = 22 + 78e^{-kt}$

iii) $T = 50$, $t = 90$

$50 = 22 + 78e^{-90k}$

$28 = 78e^{-90k}$

$e^{-90k} = \frac{14}{39}$

$-90k \doteq \log_e\left(\frac{14}{39}\right)$

$k = -\frac{1}{90}\log_e\left(\frac{14}{39}\right)$

iv) Rate is

$\frac{dT}{dt} = -k(T - A)$

$\frac{dT}{dt} = \frac{1}{90}\log_e\left(\frac{14}{39}\right)(50 - 22)$

$= \frac{1}{90}\log_e\left(\frac{14}{39}\right) \times 28$

$\doteq -0.3187\ldots$

$= -0.32$ C°/min

(to 2 sig figs)

Question 5

a) $\left[\frac{\sin^3 x}{3}\right]_0^{\frac{\pi}{4}}$

$= \frac{1}{3}\left(\sin^3\frac{\pi}{4} - \sin^3 0\right)$

$= \frac{1}{3} \times \left(\frac{1}{\sqrt{2}}\right)^3$

$= \frac{1}{3} \times \frac{1}{2\sqrt{2}}$

$= \frac{1}{6\sqrt{2}}$ (or $\frac{\sqrt{2}}{12}$)

b) $\frac{dV}{dt} = 5$ cm³/s

now $\frac{dV}{dt} = \frac{dV}{dr} \times \frac{dr}{dt}$

$V = \frac{4}{3}\pi r^3$

$\frac{dV}{dr} = 4\pi r^2$

$\therefore 5 = 4\pi r^2 \times \frac{dr}{dt}$

$\frac{dr}{dt} = \frac{5}{4\pi r^2}$

$\frac{dA}{dt} = \frac{dA}{dr} \times \frac{dr}{dt}$

$A = 4\pi r^2$

$\frac{dA}{dt} = 8\pi r$

$\frac{dA}{dt} = 8\pi r \times \frac{dr}{dt}$

$= 8\pi r \times \frac{5}{4\pi r^2}$

$= \frac{10}{r}$

when $r = 20$ cm

$\frac{dA}{dt} = \frac{10}{20} = \frac{1}{2}$

∴ rate at which surface area is increasing is $\frac{1}{2}$ cm²/s

c) i) $x = 4 + \sqrt{3}\cos 3t - \sin 3t$

$\dot{x} = -3\sqrt{3}\sin 3t - 3\cos 3t$

$\ddot{x} = -9\sqrt{3}\cos 3t + 9\sin 3t$

$\ddot{x} = -9(\sqrt{3}\cos 3t - \sin 3t + 4) + 36$

$\ddot{x} = -9x + 36$

$= -9(x-4)$

which is SHM, $n=3$, centre is 4 cm

$\sqrt{3}\cos 3t - \sin 3t = R\cos 3t\cos\alpha - R\sin 3t\sin\alpha$

Equating both sides:

$R\cos\alpha = \sqrt{3}$ ①

$R\sin\alpha = 1$ ②

② ÷ ① $\tan\alpha = \frac{1}{\sqrt{3}}$

$\alpha = \tan^{-1}\left(\frac{1}{\sqrt{3}}\right) = \frac{\pi}{6}$

$R = \sqrt{(\sqrt{3})^2 + 1^2} = \sqrt{4} = 2$

∴ $2\cos\left(3t + \frac{\pi}{6}\right)$

iii) so, $x = 4 + 2\cos\left(3t + \frac{\pi}{6}\right)$

since $-1 \le \cos\left(3t + \frac{\pi}{6}\right) \le 1$

then x can be between $(4+2)$ cm and $(4-2)$ cm

ie $2 \le x \le 6$

the centre is $x = 4$

∴ amplitude is 2 cm

iv) solve:

$4 + 2\cos\left(3t + \frac{\pi}{6}\right) = 4$

$2\cos\left(3t + \frac{\pi}{6}\right) = 0$

$\cos\left(3t + \frac{\pi}{6}\right) = 0$

$3t + \frac{\pi}{6} = \frac{\pi}{2}, \frac{3\pi}{2}, \ldots$

1st time

∴ $3t + \frac{\pi}{6} = \frac{\pi}{2}$

$3t = \frac{\pi}{3}$

$t = \frac{\pi}{9}$

∴ particle first passes through $x = 4$ after $t = \frac{\pi}{9}$ sec

Question 6

a) Show that $7^n + 2 = 3N$ where N and n are integers both ≥ 1

Prove true for $n = 1$

LHS $= 7^1 + 2 = 9 = 3 \times 3$

$=$ RHS for $N = 3$

∴ statement true for $n = 1$

Assume true for $n = k$ where k is an integer > 1

ie $7^k + 2 = 3N$

now prove true for $n = k+1$

$7^{k+1} + 2 = 3M$ where M is a positive integer

LHS $= 7^{k+1} + 2$

$= 7 \times 7^k + 2$

$= 7 \times (3N - 2) + 2$ (from assumption)

$= 21N - 14 + 2$

$= 21N - 12$

$= 3(7N - 4)$

$= 3M$ where $M = 7N - 4$

$=$ RHS

∴ statement is true for $n = k+1$

∴ statement is true for $n = 1$, $n = k$ and $n = k+1$

∴ it is true for all positive integers n

b) i) $f(x) = \frac{2x+1}{x-1}$

vertical asymptote $x = 1$ (as $x - 1 \ne 0$)

horizontal asymptote

$\lim_{x\to\infty} \frac{2x+1}{x-1} = \lim_{x\to\infty} \frac{\frac{2x}{x} + \frac{1}{x}}{\frac{x}{x} - \frac{1}{x}}$

$= \lim_{x\to\infty} \frac{2 + \frac{1}{x}}{1 - \frac{1}{x}}$

$= \frac{2+0}{1-0}$

$= 2$

∴ $y = 2$ is horiz. asymptote

ii) Domain of $f^{-1}(x)$ is the same as range of $f(x)$

∴ $x \ne 2$

iii) Sketch $y = f^{-1}(x)$

D: $x \ne 2$

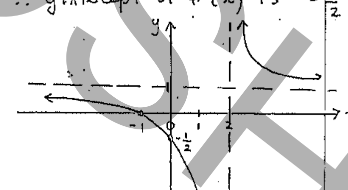
R: $y \ne 1$

for $y = f(x)$ the y-intercept is -1

∴ x intercept of $f^{-1}(x)$ is -1

for $y = f(x)$ the x-intercept is $-\frac{1}{2}$

∴ y intercept of $f^{-1}(x)$ is $-\frac{1}{2}$



) $\frac{1}{2}V^2 = \int -e^{-\frac{x}{2}}\,dx$

) $\frac{1}{2}V^2 = -\frac{e^{-\frac{x}{2}}}{-\frac{1}{2}} + C$

$\frac{1}{2}V^2 = 2e^{-\frac{x}{2}} + C$

$=0,\ v=2$

$2 = 2 + C$

$\therefore C = 0$

$\therefore V^2 = 4e^{-\frac{x}{2}}$

) $V = \pm\sqrt{4e^{-\frac{x}{2}}}$

$V = \pm 2e^{-\frac{x}{4}}$

at $x=0,\ v=2$ $\therefore$ take positive V

$V = 2e^{-\frac{x}{4}}$

$\frac{dx}{dt} = 2e^{-\frac{x}{4}} = \frac{2}{e^{\frac{x}{4}}}$

$\therefore \int \frac{e^{\frac{x}{4}}}{2}\,dx = \int dt$

$\frac{1}{2}\left(\frac{e^{\frac{x}{4}}}{\frac{1}{4}}\right) = t + K$

$\frac{1}{2} \times 4e^{\frac{x}{4}} = t + K$

$2e^{\frac{x}{4}} = t + K$

$t=0,\ x=0$ $\quad 2 = 0 + K$

$\therefore K = 2$

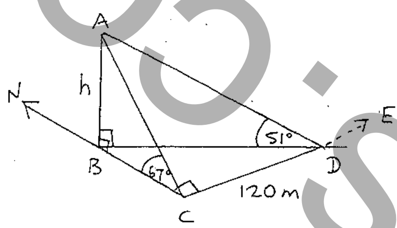
$2e^{\frac{x}{4}} = t + 2$

$e^{\frac{x}{4}} = \frac{t+2}{2}$

$\frac{x}{4} = \log_e\left(\frac{t+2}{2}\right)$

$\therefore x = 4\log_e\left(\frac{t+2}{2}\right)$

Question 7



i) $\tan 67° = \frac{h}{BC}$ $\qquad \tan 51° = \frac{h}{BD}$

$BC = \frac{h}{\tan 67°}$ $\qquad BD = \frac{h}{\tan 51°}$

$BC = h\tan 23°$ $\qquad BD = h\tan 39°$

In the base $\triangle BDC$:

use Pythagoras' Theorem since $\angle BCD = 90°$

$BD^2 = BC^2 + 120^2$

$(h\tan 39)^2 = (h\tan 23)^2 + 120^2$

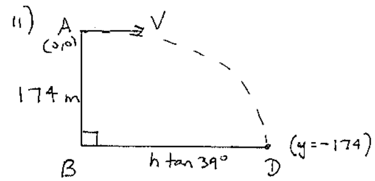
$h^2(\tan^2 39 - \tan^2 23) = 120^2$

$h^2 = \frac{120^2}{\tan^2 39 - \tan^2 23}$

$h = \frac{120}{\sqrt{\tan^2 39 - \tan^2 23}}$

$\therefore h = 174$ m (nearest m)

ii)



horizontal projection $\therefore \theta = 0$

$x = Vt\cos\theta$, $y = Vt\sin\theta - 5t^2$

$\theta = 0$

$x = Vt$, $y = -5t^2$

At point D, $y = -174$

$-174 = -5t^2$

$t^2 = \frac{174}{5}$

$t = \sqrt{\frac{174}{5}} \doteqdot 5.9$ sec

then $x = Vt$

h = 174 → $h\tan 39° = V \times 5.9$

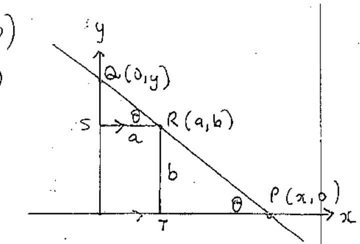
$V = \frac{174 \times \tan 39°}{5.9}$

$= 23.8817\ldots$

$\therefore V = 23.9$ m/s

b)

i)



i) length $PQ = QR + RP$

In $\triangle RTP$: $\sin\theta = \frac{b}{RP}$

$\therefore RP = \frac{b}{\sin\theta}$

In $\triangle QRS$: $\cos\theta = \frac{a}{RQ}$

$RQ = \frac{a}{\cos\theta}$

$\therefore PQ = \frac{a}{\cos\theta} + \frac{b}{\sin\theta}$

ii) let $L = \frac{a}{\cos\theta} + \frac{b}{\sin\theta}$

ie $L = a\sec\theta + b\operatorname{cosec}\theta$

to minimise length, solve $\frac{dL}{d\theta} = 0$ for θ

$L' = a\sec\theta\tan\theta + -b\operatorname{cosec}\theta\cot\theta$

$= a \times \frac{1}{\cos\theta} \times \frac{\sin\theta}{\cos\theta} - b \times \frac{1}{\sin\theta} \times \frac{\cos\theta}{\sin\theta}$

$= \frac{a\sin\theta}{\cos^2\theta} - \frac{b\cos\theta}{\sin^2\theta} = 0$ for min

$\therefore \frac{a\sin\theta}{\cos^2\theta} = \frac{b\cos\theta}{\sin^2\theta}$

$a\sin^3\theta = b\cos^3\theta$

$\therefore \tan^3\theta = \frac{b}{a}$

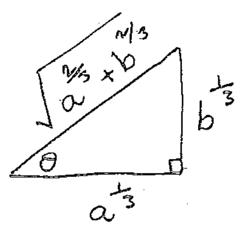
$\tan\theta = \sqrt[3]{\frac{b}{a}} = \frac{b^{\frac{1}{3}}}{a^{\frac{1}{3}}}$

$\theta = \tan^{-1}\left(\sqrt[3]{\frac{b}{a}}\right)$

$\therefore \theta = \tan^{-1}\left(\sqrt[3]{\frac{b}{a}}\right)$ gives the minimum length of PQ

(* it is a min. length since there is no max. value for PQ, since as $\theta \to 0$, length $PQ \to \infty$)

now, since $\tan\theta = \frac{b^{\frac{1}{3}}}{a^{\frac{1}{3}}}$



use Pythagoras

$\therefore \sin\theta = \frac{b^{\frac{1}{3}}}{\sqrt{a^{2/3}+b^{2/3}}}$

$\cos\theta = \frac{a^{\frac{1}{3}}}{\sqrt{a^{2/3}+b^{2/3}}}$

$\therefore$ Minimum length of PQ:

$PQ = \frac{a}{\cos\theta} + \frac{b}{\sin\theta}$

$= \frac{a}{\frac{a^{\frac{1}{3}}}{\sqrt{a^{2/3}+b^{2/3}}}} + \frac{b}{\frac{b^{\frac{1}{3}}}{\sqrt{a^{2/3}+b^{2/3}}}}$

$= a^{2/3}\sqrt{a^{2/3}+b^{2/3}} + b^{2/3}\sqrt{a^{2/3}+b^{2/3}}$

$PQ = a^{\frac{2}{3}}\sqrt{a^{\frac{2}{3}}+b^{\frac{2}{3}}} + b^{\frac{2}{3}}\sqrt{a^{2/3}+b^{2/3}}$

$= \sqrt{a^{2/3}+b^{2/3}}\left(a^{2/3}+b^{2/3}\right)$

$= \left(a^{2/3}+b^{2/3}\right)^{\frac{1}{2}}\left(a^{2/3}+b^{2/3}\right)$

$= \left(a^{2/3}+b^{2/3}\right)^{3/2}$

!! whew!

$\therefore$ min length of PQ is equal to $\left(a^{2/3}+b^{2/3}\right)^{3/2}$