Marks

Question 1 (12 marks) (Use a separate writing booklet)

(a) Express $\dfrac{4\sqrt{13}}{\sqrt{13}-3}$ in simplest form. 2

Solution: $\dfrac{4\sqrt{13}}{\sqrt{13}-3} \times \dfrac{\sqrt{13}+3}{\sqrt{13}+3} = \dfrac{4(13+3\sqrt{13})}{13-9}$,
$= 13+3\sqrt{13}$.

(b) The height h at time t of a particle projected vertically upwards from the ground is given by $h = 32t - 16t^2$. Find the greatest height reached. 2

Solution: $\dot{h} = 32 - 32t$,
$= 0$ when $t = 1$.
$\therefore$ the greatest height is 16.

(c) Given that $\cos(A-B) = \cos A \cos B + \sin A \sin B$, show that $\sin(A+B) = \sin A \cos B + \cos A \sin B$. 2

Solution: $\cos(90^\circ - \overline{A+B}) = \cos(\overline{90^\circ - A} - B)$,
$= \cos(90^\circ - A)\cos B + \sin(90^\circ - A)\sin B$,
but $\cos(90^\circ - \theta) = \sin\theta$, and $\sin(90^\circ - \theta) = \cos\theta$,
$\therefore\ \sin(A+B) = \sin A \cos B + \cos A \sin B$.

(d) (i) Find the point of intersection of $y = x^2$ and $y = \frac{1}{x}$. 1

Solution: $x^2 = \frac{1}{x}$,
$x^3 = 1$,
$x = 1$,
$y = 1$.
The curves intersect at (1, 1).

(ii) Find the acute angle, correct to the nearest minute, between the two curves at the point of intersection. 2

Solution: If $y = x^2$, and if $y = \dfrac{1}{x}$,
$y' = 2x$, $y' = -\dfrac{1}{x^2}$,
$= 2$ at (1, 1), $= -1$ at (1, 1).
The angle at the intersection $= \tan^{-1}\left|\dfrac{2 - {}^{-}1}{1 + 2\times(-1)}\right|$,
$= \tan^{-1}(3)$,
$= 71^\circ 34'$.

(e) Solve $\left(x+\frac{1}{x}\right)^2 - 5\left(x+\frac{1}{x}\right) + 6 = 0.$ 3

Solution: Put $y = \left(x+\frac{1}{x}\right)$, then
$$\begin{aligned} y^2 - 5y + 6 &= 0,\\ (y-3)(y-2) &= 0,\\ \therefore y &= 3 \text{ or } 2. \end{aligned}$$

So
$$\begin{aligned} \left(x+\frac{1}{x}\right) &= 3, \\ i.e.\ x^2 - 3x + 1 &= 0, \\ x &= \frac{3 \pm \sqrt{9-4}}{2}, \end{aligned}$$
and
$$\begin{aligned} \left(x+\frac{1}{x}\right) &= 2, \\ x^2 - 2x + 1 &= 0, \\ (x-1)^2 &= 0, \end{aligned}$$
$$\therefore x = 1,\ \frac{3 \pm \sqrt{5}}{2}.$$

Marks

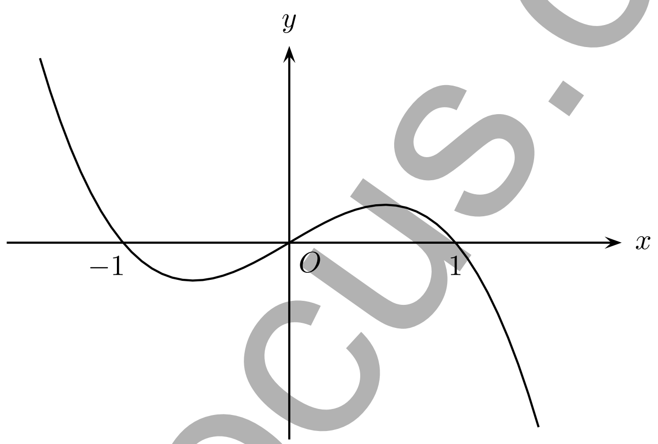
Question 2 (12 marks) (Use a separate writing booklet)

(a) Determine the numerical values of a and b such that $\cos 3\theta = a\cos^3\theta + b\cos\theta$ is an identity in θ. 3

Solution:
$$\begin{aligned}\cos(2\theta+\theta) &= \cos 2\theta.\cos\theta - \sin 2\theta.\sin\theta,\\ &= \big(2\cos^2\theta - 1\big)\cos\theta - 2\sin\theta\cos\theta.\sin\theta,\\ &= 2\cos^3\theta - \cos\theta - 2\big(1-\cos^2\theta\big)\cos\theta,\\ &= 4\cos^3\theta - 3\cos\theta.\\ \therefore\ a &= 4 \text{ and } b = -3.\end{aligned}$$

(b) Sketch $f(x) = (x+1)x(1-x)$. 2

Solution:



(c) Eliminate t from this pair of parametric equations 3

$$\left.\begin{aligned} x &= 3\tan t \\ y &= 2\sec t \end{aligned}\right\}$$

and so form the corresponding Cartesian equation.

Solution:
$$\begin{aligned}\tan t &= \frac{x}{3} \quad\text{and}\quad \sec t = \frac{y}{2},\\ \therefore \frac{x^2}{9} + 1 &= \frac{y^2}{4},\\ \textit{i.e. } 4x^2 - 9y^2 + 36 &= 0.\end{aligned}$$

(d) In a 12-horse race five of the horses really have hardly any chance of getting a place and for this question may be disregarded.

(i) In how many ways can the first three places be filled from the remaining horses? 1

Solution: $7 \times 6 \times 5$ or ${}^7P_3 = 210$.

(ii) If these horses are all about equally likely to gain places, what approximately are the chances of picking the trifecta (*i.e.* the first three in the correct order)? 1

Solution: $\dfrac{1}{210}$.

(iii) The quinella is the first two horses without regard to order. Neglecting the five no-hopers, what is the probability of correctly picking the quinella? 2

Solution: $\dfrac{2}{7 \times 6}$ or $\dfrac{1}{{}^7C_2} = \dfrac{1}{21}$.

Marks

Question 3 (12 marks) (Use a separate writing booklet)

(a) (i) By examining its first and second derivatives, find any stationary points or points of inflexion on the curve $y = \sqrt[3]{x} - x$. **4**

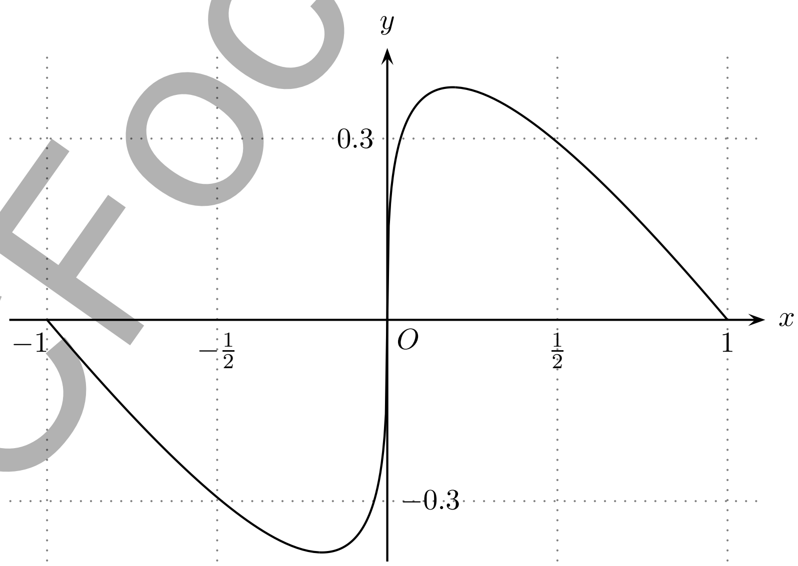
Solution:

$$\begin{aligned} y &= x^{\frac{1}{3}} - x, \\ y' &= \tfrac{1}{3}x^{-\frac{2}{3}} - 1, \\ &= \text{undefined when } x = 0, \\ &= 0 \text{ when } \frac{1}{3x^{\frac{2}{3}}} = 1, \\ &\qquad i.e.\ x = \pm\frac{1}{\sqrt{27}} \approx \pm 0.192, \\ &\qquad \text{and } y \approx \pm 0.385. \\ y'' &= -\tfrac{2}{9}x^{-\frac{5}{3}}, \\ &\neq 0 \text{ for any } x, \\ &= \text{undefined when } x = 0, \\ &\approx -3.464 \text{ when } x \approx 0.192, \\ &\approx 3.464 \text{ when } x \approx -0.192. \end{aligned}$$

Thus we have a maximum at about (0.192, 0.385) and a minimum at about (−0.192, −0.385), as well as a vertical point of inflexion at (0, 0).

(ii) Hence sketch $y = \sqrt[3]{x} - x$ over $[-1, 1]$. **2**

Solution:



(b) Prove by induction that **3**

$$\frac{1}{x^n(x-1)} = \frac{1}{x-1} - \frac{1}{x} - \frac{1}{x^2} - \cdots - \frac{1}{x^n}$$

for all positive integers n and $x \neq 0, 1$.

Solution: Test for $n = 1$, L.H.S. $= \dfrac{1}{x(x-1)}$, R.H.S. $= \dfrac{1}{x-1} - \dfrac{1}{x}$,

$$= \frac{x-(x-1)}{x(x-1)},$$

$$= \text{L.H.S.}$$

Assume true for some particular $n = k$, say;

$$i.e.\ \frac{1}{x^k(x-1)} = \frac{1}{x-1} - \frac{1}{x} - \frac{1}{x^2} - \cdots - \frac{1}{x^k},$$

then test for $n = k+1$;

$$i.e.\ \frac{1}{x^{k+1}(x-1)} = \frac{1}{x-1} - \frac{1}{x} - \frac{1}{x^2} - \cdots - \frac{1}{x^{k+1}}.$$

Now R.H.S. $= \dfrac{1}{x^k(x-1)} - \dfrac{1}{x^{k+1}}$ (using the assumption),

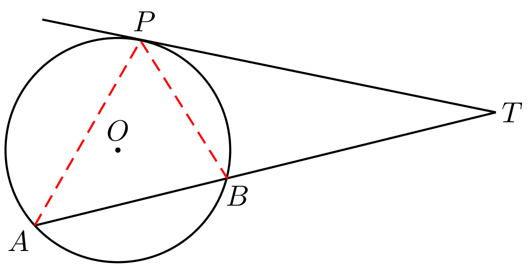
$$= \frac{x^{k+1} - x^k(x-1)}{x^k x^{k+1}(x-1)},$$

$$= \frac{x^{k+1} - x^{k+1} + x^k}{x^k x^{k+1}(x-1)},$$

$$= \frac{1}{x^{k+1}(x-1)},\ i.e.\ = \text{L.H.S.}$$

Hence the statement is true by the principle of Mathematical Induction.

(c) Prove that $TP^2 = TA \times TB$. **3**



Solution: Construction: draw PA and PB.

Proof: $\angle TPB = \angle PAB$ ($\angle$ at chord of contact equals $\angle$ in alt. segment),

$\angle PTB = \angle ATP$ (common),

$\triangle PTB \mathbin{|||} \triangle ATP$ (equiangular),

$\dfrac{PT}{AT} = \dfrac{TB}{TP}$ (corresp. sides of similar $\triangle$s),

$i.e.\ TP^2 = TA \times TB.$

Marks

Question 4 (12 marks) (Use a separate writing booklet)

(a) Use one application of Newton's method to estimate that root of $x^3-6x^2+24=0$ which lies near $x=3$. 2

> **Solution:**
> $$\begin{aligned} x_1 &\approx x_0 - \frac{f(x_0)}{f'(x_0)}, \\ &\approx x_0 - \frac{x_0^3 - 6x_0^2 + 24}{3x_0^2 - 12x_0}, \\ &\approx 3 - \frac{3^3 - 6\times 3^2 + 24}{3\times 3^2 - 12\times 3}, \\ &\approx 2\tfrac{2}{3}. \end{aligned}$$

(b) Find

(i) $\displaystyle\int \sin^2\left(\frac{\theta}{2}\right)\,d\theta$, 2

> **Solution:** Note: $\cos 2A = 1 - 2\sin^2 A$,
> $$\text{so } \sin^2 A = \frac{1-\cos 2A}{2}.$$
> $$\begin{aligned} \int \sin^2\left(\frac{\theta}{2}\right)\,d\theta &= \tfrac{1}{2}\int (1-\cos\theta)\,d\theta, \\ &= \tfrac{1}{2}(\theta - \sin\theta) + c. \end{aligned}$$

(ii) $\displaystyle\int \tan\phi\sec^2\phi\,d\phi$. (Let $u = \tan\phi$.) 2

> **Solution:**
> $$\begin{aligned} \frac{du}{d\phi} &= \sec^2\phi, \\ \text{so } \int \tan\phi\sec^2\phi\,d\phi &= \int u \times \frac{du}{d\phi}\times d\phi, \\ &= \int u\,du, \\ &= \frac{u^2}{2} + c, \\ &= \frac{\tan^2\phi}{2} + c. \end{aligned}$$

(c) (i) State the domain and range of the function $f(x) = x\sin^{-1}\left(x^2\right)$. 2

Solution: Clearly the first $x \in \mathbb{R}$, and within the parentheses $0 \leqslant x^2 \leqslant 1$, so the domain: $-1 \leqslant x \leqslant 1$.
When $x = -1$, $\sin^{-1} 1 = \frac{\pi}{2}$, and when $x = 1$, $\sin^{-1} 1 = \frac{\pi}{2}$.
Hence the range: $-\frac{\pi}{2} \leqslant f(x) \leqslant \frac{\pi}{2}$.

(ii) Determine the derivative of $x\sin^{-1}\left(x^2\right)$ and describe the behaviour of the function in the neighbourhood of: 4

(α) $x = 0$; and

(β) $x = 1$.

Solution:
$$f'(x) = 1 \times \sin^{-1}\left(x^2\right) + \frac{x \times 2x}{\sqrt{1-(x^2)^2}},$$
$$= \sin^{-1}\left(x^2\right) + \frac{2x^2}{\sqrt{1-x^4}}.$$

(α) Now when $x = 0$, $\sin^{-1} 0 = 0$, and $\dfrac{2 \times 0^2}{\sqrt{1-0^4}} = 0$,
so the function becomes horizontal (*i.e.* has a stationary point) at $x = 0$.

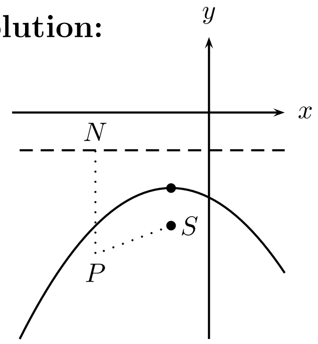
(β) Near $x = 1$, $\lim\limits_{x\to 1}\{f'(x)\} = \dfrac{\pi}{2} + \infty$, *i.e.* the slope becomes undefined and the curve becomes vertical.

Marks

Question 5 (12 marks) (Use a separate writing booklet)

(a) A parabola in the Cartesian plane has its vertex at $(-1, -2)$ and its focus at $(-1, -3)$. Derive an inequality in x and y which is satisfied by the coördinates of a point $P\,(x,\, y)$ if and only if P is closer to the focus of the parabola than it is to the directrix of the parabola. 3

Solution:



From a rough sketch, we see that

$$\begin{aligned}(y - {}^{-}1)^2 &> (x - {}^{-}1)^2 + (y - {}^{-}3)^2,\\ y^2 + 2y + 1 &> x^2 + 2x + 1 + y^2 + 6y + 9,\\ -4y - 8 &> (x+1)^2,\\ \therefore (x+1)^2 &< -4(y+2).\end{aligned}$$

(b) Find

(i) $\displaystyle\int_0^4 \frac{x\,dx}{\sqrt{9+x^2}}$ (use the substitution $u = 9 + x^2$), 3

Solution:

$$\begin{aligned}\frac{du}{dx} &= 2x,\\ \text{when } x &= 0, \quad u = 9,\\ \text{when } x &= 4, \quad u = 25.\\ \int_0^4 \frac{x\,dx}{\sqrt{9+x^2}} &= \tfrac{1}{2}\int_0^4 \frac{2x\,dx}{\sqrt{9+x^2}},\\ &= \tfrac{1}{2}\int_9^{25} \frac{du}{dx}.\frac{dx}{\sqrt{u}},\\ &= \tfrac{1}{2}\int_9^{25} u^{-\frac{1}{2}}\,du,\\ &= \tfrac{1}{2}\Big[2u^{\frac{1}{2}}\Big]_9^{25},\\ &= 5 - 3,\\ &= 2.\end{aligned}$$

(ii) $\displaystyle\int \frac{x\,dx}{\sqrt{9+x^2}}$ (use the substitution $u = \sqrt{9+x^2}$). 3

Solution:
$$\begin{aligned}
\frac{du}{dx} &= 2x \times \tfrac{1}{2}(9+x^2)^{-\frac{1}{2}},\\
&= \frac{x}{\sqrt{9+x^2}},\\
\int \frac{x\,dx}{\sqrt{9+x^2}} &= \int \frac{du}{dx} \times dx,\\
&= \int du,\\
&= u + c,\\
&= \sqrt{9+x^2} + c.
\end{aligned}$$

(c) By considering the derivative of $\tan^{-1}\left(\dfrac{x}{a}\right)$, 3
show that
$$\int \frac{dx}{a^2+x^2} = \frac{1}{a}\tan^{-1}\frac{x}{a} + c.$$

Solution: Put
$$\begin{aligned}
y &= \tan^{-1}\frac{x}{a},\\
x &= a\tan y,\\
\frac{dx}{dy} &= a\sec^2 y,\\
&= a(1+\tan^2 y),\\
&= \frac{a^2 + a^2\tan^2 y}{a},\\
&= \frac{a^2+x^2}{a},\\
\therefore \frac{dy}{dx} &= \frac{a}{a^2+x^2}.\\
\text{So } \int \frac{dy}{dx}.dx &= \int \frac{a\,dx}{a^2+x^2},\\
y &= \int \frac{a\,dx}{a^2+x^2},\\
\int \frac{dx}{a^2+x^2} &= \frac{1}{a}\tan^{-1}\frac{x}{a} + c.
\end{aligned}$$

Marks

Question 6 (12 marks) (Use a separate writing booklet)

(a) Use the remainder theorem to find one factor of $x(x+1)-a(a+1)$. By division, or otherwise, find the other factor. 3

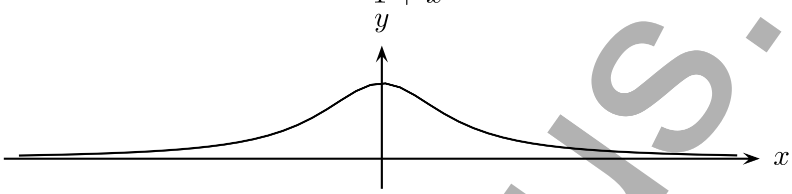
Solution: Put
$$\begin{aligned} P(x) &= x(x+1)-a(a+1), \\ P(a) &= a^2+a-a^2-a, \\ &= 0. \end{aligned}$$

Using Horner's method:

	1	1	$-(a^2+a)$
a		a	a^2+a
	1	$(a+1)$	0

So $x(x+1)-a(a+1)=(x-a)(x+a+1)$.

(b) Consider the function $y=\dfrac{1}{1+x^2}$ sketched below:



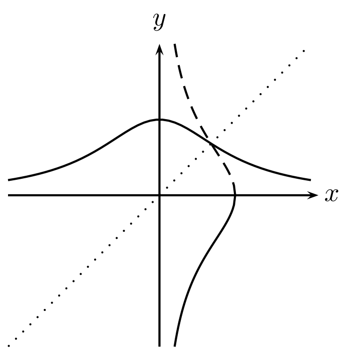
(i) Show that, by a suitable restriction on the domain of this function, a monotonic increasing inverse function can be found. 3

Solution: To find an inverse, we interchange x and y:
$$\begin{aligned} x &= \frac{1}{1+y^2}, \\ y^2+1 &= \tfrac{1}{x}, \\ y &= \pm\sqrt{\tfrac{1}{x}-1} \end{aligned}$$

To decide which of the two to choose there are two methods:
The first is a simple sketch—

This shows that we should select the negative option

$$y=-\sqrt{\frac{1}{x}-1}.$$



The second option is to differentiate one of the choices and examine its sign:

Taking $y = \sqrt{\frac{1}{x} - 1}$,

$$\begin{aligned} \frac{dy}{dx} &= \frac{-1}{x^2}.\frac{1}{2}.\frac{1}{\sqrt{\frac{1}{x}-1}}, \\ &= -\frac{1}{2x\sqrt{x-x^2}}, \\ &< 0 \text{ when } 0 < x < 1. \end{aligned}$$

As the positive option is always monotonic decreasing, we must choose the negative option

$$y = -\sqrt{\frac{1}{x} - 1}.$$

(ii) Write the domain and range of the inverse function. 1

Solution: Domain: $0 < x \leqslant 1$.
Range: $y \leqslant 0$.

(c) A typical application of the Verhulst logistic equation is a common model of population growth, which states that:

- the rate of reproduction is proportional to the existing population, all else being equal;
- the rate of reproduction is proportional to the amount of available resources, all else being equal. Thus the second term models the competition for available resources, which tends to limit the population growth.

Letting P represent population size (N is often used in ecology instead) and t represent time, this model is formalised by the differential equation:

$$\frac{dP}{dt} = rP\left(1 - \frac{P}{K}\right),$$

where the constant r defines the growth rate and K is the carrying capacity. In ecology, species are sometimes referred to as r-strategist or K-strategist depending upon the selective processes that have shaped their life history strategies.

(i) Show by differentiation that the solution to the equation (with P_0 being the initial population) is [3]

$$P(t) = \frac{KP_0e^{rt}}{K + P_0(e^{rt} - 1)}.$$

Solution:
$$\begin{aligned}
\frac{dP}{dt} &= \frac{\{K + P_0(e^{rt} - 1)\} \times rKP_0e^{rt} - KP_0e^{rt} \times rP_0e^{rt}}{\{K + P_0(e^{rt} - 1)\}^2}, \\
&= \left(\frac{r \times KP_0e^{rt}}{K + P_0(e^{rt} - 1)}\right)\left(\frac{(K + P_0(e^{rt} - 1)) - P_0e^{rt}}{K + P_0(e^{rt} - 1)}\right), \\
&= rP\left(1 - \frac{P_0e^{rt}}{K + P_0(e^{rt} - 1)} \times \frac{K}{K}\right), \\
&= rP\left(1 - \frac{P}{K}\right).
\end{aligned}$$

(ii) Confirm that the final population will equal the carrying capacity, *i.e.* that [1]

$$\lim_{t\to\infty} P(t) = K.$$

Solution:
$$\begin{aligned}
\lim_{t\to\infty} P(t) &= \lim_{t\to\infty}\left\{\frac{KP_0e^{rt}}{K + P_0(e^{rt} - 1)}\right\}, \\
&= \lim_{t\to\infty}\left\{\frac{KP_0}{Ke^{-rt} + P_0(1 - e^{-rt})}\right\}, \\
&= \frac{KP_0}{Ke^{-\infty} + P_0(1 - e^{-\infty})}, \\
&= \frac{KP_0}{K \times 0 + P_0(1 - 0)}, \\
&= K \text{ as required.}
\end{aligned}$$

(iii) If $r = 1.25$; $K = 25000$; $P_0 = 10$, find P after 5 years. [1]

Solution:
$$\begin{aligned}
P(5) &= \frac{25000 \times 10 \times e^{1.25\times5}}{25000 + 10(e^{1.25\times5} - 1)}, \\
&= 4292.431411 \text{ by calculator.}
\end{aligned}$$

i.e. The population will be about 4290.

Marks

Question 7 (12 marks) (Use a separate writing booklet)

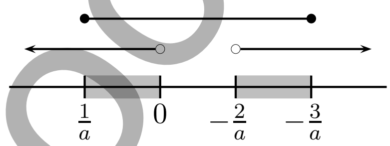
(a) $P(14,\,18)$ divides the interval AB externally in the ratio $3:2$. If $B=(4,\,8)$, find the coördinates of A. **2**

> **Solution:**
> $$\begin{aligned} 14 &= \frac{-2x+3\times 4}{3+{}^-2}, & 18 &= \frac{-2y+3\times 8}{3+{}^-2},\\ &= -2x+12, & &= -2y+24,\\ -2x &= 2, & -2y &= -6,\\ x &= -1, & y &= 3. \end{aligned}$$
> $\therefore$ The coördinates of A are $(-1,\,3)$.

(b) Given that $a<0$, find the solution to the inequalities **4**

$$1<|ax+1|\leqslant 2,$$

leaving your answers in terms of a.

> **Solution:** $|ax+1|>1$, and $|ax+1|\leqslant 2$,
> $$\begin{aligned} &ax+1>1 \quad\text{or}\quad ax+1<-1, & -2\leqslant ax+1 &\leqslant 2,\\ &ax>0 \qquad\quad\;\; ax<-2, & -3\leqslant ax &\leqslant 1,\\ &x<0 \qquad\quad\;\;\; x>-\frac{2}{a}, & -\frac{3}{a}\geqslant x,\ x &\geqslant \frac{1}{a}. \end{aligned}$$
> Graphing these gives
>
> 
>
>
> $\therefore$ Solution is $\dfrac{1}{a}\leqslant x<0$ or $-\dfrac{2}{a}<x\leqslant -\dfrac{3}{a}$.

(c) The acceleration $a\,\text{cm/s}^2$ of a particle P is given by $a = 18x(x^2+1)$, where x cm is the displacement of P at t seconds. Initially P starts from the origin with velocity 3 cm/s.

(i) Show that $v = 3(x^2+1)$, **2**

Solution:
$$\begin{aligned}
\frac{d}{dx}\left(\tfrac{1}{2}v^2\right) &= 18x(x^2+1),\\
\tfrac{1}{2}v^2 &= \frac{18x^4}{4} + \frac{18x^2}{2} + c,\\
&= \tfrac{9}{2}x^4 + 9x^2 + c.\\
\text{Initially } x=0,\ v=3; \Rightarrow \frac{9}{2} &= \frac{9\times 0}{2} + 9\times 0 + c,\\
i.e.\ v^2 &= 9x^4 + 18x^2 + 9,\\
&= 9(x^2+1)^2,\\
\therefore v &= 3(x^2+1) \text{ taking positive } v \text{ as } v_0 = 3.
\end{aligned}$$

(ii) and show that $x = \tan 3t$. **2**

Solution:
$$\begin{aligned}
v &= \frac{dx}{dt},\\
\frac{dt}{dx} &= \frac{1}{3(x^2+1)},\\
t &= \tfrac{1}{3}\tan^{-1}x + c.\\
\text{Initially } x=0,\ t=0; \Rightarrow 0 &= \tfrac{1}{3}\times 0 + c,\\
i.e.\ \tan^{-1}x &= 3t,\\
x &= \tan 3t.
\end{aligned}$$

(iii) Determine its velocity and displacement after $\frac{\pi}{12}$ s. **2**

Solution: When $t = \frac{\pi}{12}$,
$$\begin{aligned}
x &= \tan\frac{\pi}{4},\\
&= 1.\\
v &= 3(1+1),\\
&= 6.
\end{aligned}$$
So the displacement is 1 cm in the positive direction from the origin and the velocity is 6 cm/s away from the origin in the positive direction.

End of Paper