

2007 THSC Mathematics Extension 1: Solutions— Question 1

1. (a) Evaluate $\lim_{x \rightarrow 0} \frac{\sin 4x}{5x}$.

1

Solution:

$$\begin{aligned}\lim_{x \rightarrow 0} \frac{\sin 4x}{5x} &= \lim_{x \rightarrow 0} \frac{\sin 4x}{4x} \times \frac{4}{5}, \\ &= \frac{4}{5} \times \lim_{x \rightarrow 0} \frac{\sin 4x}{4x}, \\ &= \frac{4}{5}.\end{aligned}$$

- (b) Calculate the acute angle (to the nearest minute) between the lines $2x + y = 4$ and $x - 3y = 6$.

2

Solution:

$$\begin{aligned}\tan \alpha &= \frac{|-2 - 1/3|}{1 + (-2) \times (1/3)}, \\ &= 7, \\ \therefore \alpha &= \tan^{-1} 7, \\ &= 81.86989765^\circ \text{ by calculator,} \\ &= 81^\circ 52' .\end{aligned}$$

- (c) i. Show that $x + 1$ is a factor of $x^3 - 4x^2 + x + 6$.

1

Solution: Putting $P(x) = x^3 - 4x^2 + x + 6$;

$$\begin{aligned}P(-1) &= -1 - 4 - 1 + 6, \\ &= 0. \\ \therefore x + 1 &\text{ is a factor.}\end{aligned}$$

- ii. Hence or otherwise factorise $x^3 - 4x^2 + x + 6$ fully.

2

Solution: Possible factors of 6 are 1, 2, 3 or 1, -2, -3.

$$\begin{aligned}P(-2) &= -8 - 16 - 2 + 6 \neq 0, \\ P(2) &= 8 - 16 + 2 + 6, \\ &= 0. \\ \therefore x^3 - 4x^2 + x + 6 &= (x + 1)(x - 2)(x - 3).\end{aligned}$$

- (d) The point $P(5, 7)$ divides the interval joining the points $A(-1, 1)$ and $B(3, 5)$ externally in the ratio $k : 1$. Find the value of k .

2

Solution:
$$\frac{5 - (-1)}{5 - 3} = \frac{k}{1},$$
$$6 = 2k,$$
$$k = 3.$$

- (e) Find the horizontal asymptote of the function $y = \frac{3x^2 - 4x + 1}{2x^2 - 1}$.

1

Solution:
$$\lim_{x \rightarrow \pm\infty} \frac{3 - 4/x + 1/x^2}{2 - 1/x^2} = \frac{3}{2}.$$
$$\therefore y = 3/2 \text{ is the horizontal asymptote.}$$

- (f) Find a primitive of $\frac{1}{\sqrt{4 - x^2}}$.

1

Solution: From the table of standard integrals,

$$\int \frac{dx}{\sqrt{4 - x^2}} = \sin^{-1} \frac{x}{2} + c.$$

- (g) Solve the equation $|x + 1|^2 - 4|x + 1| - 5 = 0$.

2

Solution: Putting $y = |x + 1|$;
 $y^2 - 4y - 5 = 0,$
 $(y - 5)(y + 1) = 0,$
 $\therefore y = 5 \text{ or } -1.$
But $|x + 1| \geq 0,$
hence $x + 1 = 5$ or $x + 1 = -5,$
so $x = 4, -6.$

(a) i) $f(x) = \frac{1}{2} \cos^{-1}\left(\frac{x}{3}\right)$

Range: $0 \leq \cos^{-1}x \leq \pi$

$y = \cos^{-1}x$ has $-1 \leq x \leq 1$ and $0 \leq y \leq \pi$

$\frac{1}{2} \times 0 \leq \frac{1}{2} \cos^{-1}\left(\frac{x}{3}\right) \leq \frac{1}{2} \times \pi$

Domain $-1 \leq \frac{x}{3} \leq 1$
 $-3 \leq x \leq 3$ ①

$0 \leq f(x) \leq \frac{\pi}{2}$ ①

ii) $f'(x) = \frac{1}{2} \times \frac{-1}{\sqrt{1 - \frac{x^2}{9}}} \times \frac{1}{3}$

$= -\frac{1}{6} \times \frac{1}{\sqrt{\frac{9-x^2}{9}}} = -\frac{1}{6} \times \frac{3}{\sqrt{9-x^2}} = -\frac{1}{2} \cdot \frac{1}{\sqrt{9-x^2}}$

So for $-3 < x < 3$, $f'(x) < 0$ always. ②

iii) when $x=0$, $f(x) = \frac{1}{2} \cos^{-1}\left(\frac{0}{3}\right)$
 $= \frac{1}{2} \cos^{-1}(0)$
 $= \frac{1}{2} \times \frac{\pi}{2} = \frac{\pi}{4}$

$f'(x) = \frac{-1}{2\sqrt{9-x^2}}$

at $x=0$, $m = \frac{-1}{2 \times 3} = -\frac{1}{6}$

$(y - y_1) = m(x - x_1)$

$(y - \frac{\pi}{4}) = -\frac{1}{6}(x - 0)$

$y = -\frac{1}{6}x + \frac{\pi}{4}$

or $\frac{1}{6}x + y - \frac{\pi}{4} = 0$

or $12 \times \frac{1}{6}x + 12y - 12 \times \frac{\pi}{4} = 0$

$2x + 12y - 3\pi = 0$ ②

(b) $y = \ln(\sin^3 x)$

$y' = \frac{1}{\sin^3 x} \times 3\sin^2 x \times \cos x \times 1$

$= \frac{3\sin^2 x \cos x}{\sin^3 x}$

$= 3 \frac{\cos x}{\sin x}$

$= 3 \cot x$ ②

(C) (i) $1 \cos x - \sqrt{3} \sin x$

$$A = \sqrt{1^2 + (-\sqrt{3})^2} = \sqrt{4} = 2$$

$$= 2 \left(\frac{1}{2} \cos x - \frac{\sqrt{3}}{2} \sin x \right)$$

$$= A (\cos x \cos \alpha - \sin x \sin \alpha)$$

$$\Rightarrow A = 2 \quad (1)$$

$$\left. \begin{array}{l} \cos \alpha = \frac{1}{2} \\ \sin \alpha = \frac{\sqrt{3}}{2} \end{array} \right\} \alpha = 60^\circ = \frac{\pi}{3} \quad (1^{st} \text{ quad } 0 < \alpha < \frac{\pi}{2})$$

$$\text{So } \cos x - \sqrt{3} \sin x = 2 \cos \left(x + \frac{\pi}{3} \right)$$

(ii) Now $\cos x - \sqrt{3} \sin x + 1 = 0$
 $\cos x - \sqrt{3} \sin x = -1$

$$2 \cos \left(x + \frac{\pi}{3} \right) = -1$$

$$\cos \left(x + \frac{\pi}{3} \right) = -\frac{1}{2}$$

$$\cos \frac{\pi}{3} = \frac{1}{2}$$

2nd / 3rd quad.

now $x + \frac{\pi}{3} = \pi - \frac{\pi}{3}$

2nd quad.

and $x + \frac{\pi}{3} = \pi + \frac{\pi}{3}$

3rd quad.

$$\text{So } x = \pi - \frac{2\pi}{3} = \frac{3\pi - 2\pi}{3} = \frac{\pi}{3} \quad \checkmark$$

(2)

and $x = \pi + \frac{\pi}{3} - \frac{\pi}{3} = \pi \quad \checkmark$

Q3 (a) (i) let $f(x) = e^x - x - 2$.

now $f(1) = e - 1 - 2 = e - 3 < 0$. (≈ -0.28)

& $f(2) = e^2 - 2 - 2 = e^2 - 4 > 0$. (≈ 3.38).

Since $f(x)$ changes sign in $1 < x < 2$ (✓)

$f(x) = 0$ has a solution in $1 < x < 2$.

(ii) now $x_1 = x_1 - \frac{f(x_1)}{f'(x_1)}$ & $f(x) = e^x - x - 2$
 $f'(x) = e^x - 1$.

$\therefore x_1 = 1.5 - \frac{f(1.5)}{f'(1.5)}$

$= 1.5 - \frac{(e^{1.5} - 1.5 - 2)}{e^{1.5} - 1}$

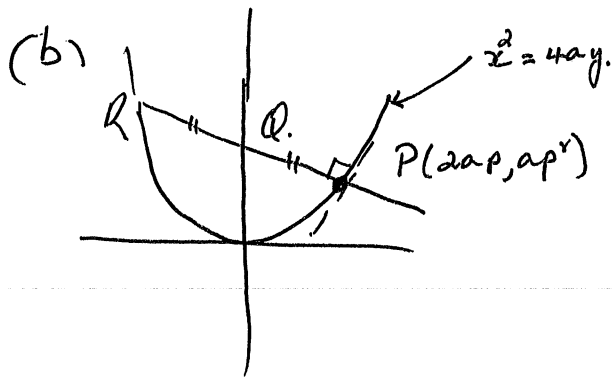
$\approx 1.5 - \frac{0.98168}{3.481689}$

$\approx 1.5 - 0.281689$

≈ 1.218

≈ 1.218

(✓✓)



(i) $y = \frac{1}{4a} x^2$

$$y' = \frac{1}{2a} x$$

$$\therefore m_T = \frac{2ap}{2a} = p$$

$$\therefore m_N = -\frac{1}{p}$$

H: $\frac{y - ap^2}{x - 2ap} = -\frac{1}{p}$

$$py - ap^3 = -x + 2ap$$

$$\boxed{x + py = 2ap + ap^3} \quad (\checkmark \checkmark)$$

(ii) Co-ords of Q. $x = 0 \quad \therefore py = 2ap + ap^3$

$$y = 2a + ap^2$$

$$\therefore Q(0, 2a + ap^2) \quad (\checkmark)$$

(iii) Q is the mid-pt of PR.

Let R be (x_1, y_1)

$$\therefore \frac{x_1 + 2ap}{2} = 0$$

$$\therefore x_1 = -2ap$$

$$\frac{y_1 + ap^2}{2} = 2a + ap^2$$

$$y_1 + ap^2 = 4a + 2ap^2$$

$$y_1 = ap^2 + 4a$$

$$\therefore R(-2ap, ap^2 + 4a) \quad (\checkmark)$$

(iii) To find the locus of R. $x = -2ap \quad \therefore p = \frac{x}{-2a}$

$$\therefore y = a\left(\frac{x}{-2a}\right)^2 + 4a$$

$$y = \frac{x^2}{4a} + 4a$$

$$4ay = x^2 + 16a^2$$

$$x^2 = 4ay - 16a^2$$

$$x^2 = 4a(y - 4a)$$

PARABOLA VERTEX
(0, 4a).

(✓✓✓)

(c)

$$\int_1^5 (2f(x) + 1) dx = 2 \int_1^5 f(x) dx + \int_1^5 1 \cdot dx$$

$$= 2 \times 3 + [x]_1^5$$

$$= 6 + (5 - 1)$$

$$= 10$$

(✓✓)

Question (1)

(a) Let $u = e^x$.

$du = e^x dx$

$\int e^{(e^x + x)} dx$

$= \int e^u \cdot e^x dx$

$= \int e^u du = e^u + C$

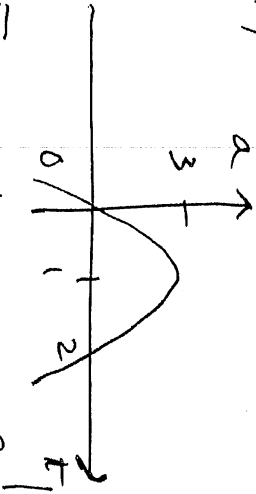
$\Rightarrow \frac{e^{e^x + x}}{e^x + C}$

(b) For $0 \leq t \leq 2$.

(i) $v = 3t^2 - t^3$

$a = \frac{dv}{dt} = \frac{6t - 3t^2}{3t(2-t)}$

(ii)



From the graph of a versus t.

The max. acceleration occurs when $t=1$

$\therefore a_{max} = 3 \times 1 = 3 \text{ m/s}^2$

(iii) Let

$d(t)$ be the total distance travelled (which is $s(t)$)

$\therefore d(t) = 2 \int_0^2 v(t) dt$

$+ \int_2^{T-2} 4 dt$

$\therefore s(1) = 2 \int_0^2 (3t^2 - t^3) dt$

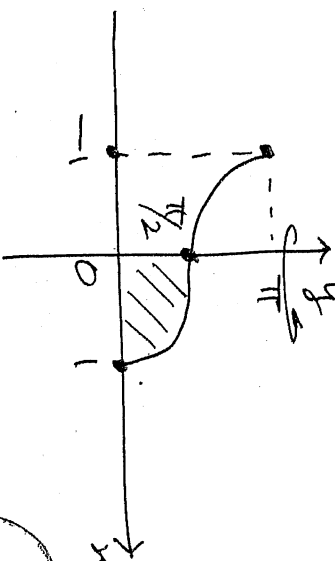
$+ [4t]_2^{T-2}$

$|s| = 2 \left[t^3 - \frac{t^4}{4} \right]_0^2 + 4(T-2)$

$\therefore |s| = 2(8-4) + 4T - 16$

$|s| = 4T - 8 \Rightarrow T = 12 \frac{1}{4}$

(c)



$y = \cos^{-1} x$

$\Rightarrow x = \cos y, \begin{cases} -1 \leq x \leq 1 \\ 0 \leq y \leq \pi \end{cases}$

$\therefore v = \pi \int_0^{\pi/2} x^2 dy$

$x = \cos y, v = \pi \int_0^{\pi/2} \cos^2 y dy$

(ii)

$v = \pi \int_0^{\pi/2} \cos^2 y dy$

but $\cos^2 y = \frac{1 + \cos 2y}{2}$

$= \frac{\pi}{2} \int_0^{\pi/2} (1 + \cos 2y) dy$

$= \frac{\pi}{2} \left[y + \frac{\sin 2y}{2} \right]_0^{\pi/2}$

$= \frac{\pi}{2} \left(\frac{\pi}{2} \right) = \frac{\pi^2}{4}$

EX1 QUESTION 5

(a) if $n=1$, $1 \times 1! = (1+1)! - 1$

$\therefore P(1)$ is true

Assume $P(k)$ is true $1 \times 1! + 2 \times 2! + \dots + k \times k! = (k+1)! - 1$

if $P(k+1)$ is $1 \times 1! + 2 \times 2! + \dots + k \times k! + (k+1)(k+1)! = (k+2)! - 1$

LHS is $(k+1)! - 1 + (k+1)(k+1)!$ using assumption

$$= (k+1)! (1 + k+1) - 1$$

$$= (k+1)! (k+2) - 1 = (k+2)! - 1 = \text{RHS}$$

$\therefore P(k+1)$ is true if $P(k)$ is true. $P(1)$ is true $\therefore$ by Mathematical Induction $\sum_{r=1}^n r \times r! = (n+1)! - 1$

(b)

$$T_{k+1} = {}^{15}C_k (2x)^{n-k} (x^{-2})^k$$

for term independent of x $n-k-2k=0$, $k=5$

$$\text{cf } {}^{15}C_5 \times 2^{10} = 3075072$$

(c) (i) $d\left(\frac{1}{2}v^2\right) = 8x(x^2+1) = 8x^3+8x$

$$\frac{1}{2}v^2 = 2x^4 + 4x^2 + C$$

$$v = -2 \quad x = 0 \quad C = 2$$

$$v^2 = 4x^4 + 8x^2 + 4 = 4(x^4 + 2x^2 + 1)$$

$$v = \pm 2(x^2+1)^2$$

(ii)

$$\text{if } \frac{dx}{dt} = 2(x^2+1)$$

$$\frac{dt}{dx} = \frac{1}{2(x^2+1)}$$

$$t = \frac{1}{2} \tan^{-1} x + C$$

$$t = 0 \quad x = 0, \quad C = 0$$

$$2t = \tan^{-1} x$$

$$x = \tan 2t$$

(iii)

$$t = \frac{\pi}{8}, \quad x = \tan \frac{\pi}{4} = 1$$

$$v = 2(1+1) \text{ from (i)}$$

$$v = 4 \text{ m/s}$$

Question 6

- (a) $\angle CBD = 60^\circ$ (alternate segment theorem)
 $\angle BCD = 90^\circ$ (angle in semicircle)
 $\therefore \angle CDB = 30^\circ$ (angle sum of triangle)
 $\therefore \angle CAB = 30^\circ$ (angles at circumference on same arc)

(b) (i) $(1+x)^n = 1 + {}^nC_1x + {}^nC_2x^2 + \dots + {}^nC_{n-1}x^{n-1} + x^n$

Differentiating with respect to x :

$$n(1+x)^{n-1} = {}^nC_1 + 2{}^nC_2x + 3{}^nC_3x^2 + \dots + n{}^nC_nx^{n-1}$$

Let $x = 1$:

$$n2^{n-1} = {}^nC_1 + 2{}^nC_2 + 3{}^nC_3 + \dots + n{}^nC_n$$

QED

- (ii) Multiplying $(1+x)^n$ by x :

$$x(1+x)^n = {}^nC_0x + {}^nC_1x^2 + {}^nC_2x^3 + \dots + {}^nC_nx^{n+1}$$

Differentiating with respect to x :

$$xn(1+x)^{n-1} + (1+x)^n = {}^nC_0 + 2{}^nC_1x + 3{}^nC_2x^2 + \dots + (n+1){}^nC_nx^n$$

Let $x = 1$:

$$n(2)^{n-1} + (2)^n = 1 + 2{}^nC_1 + 3{}^nC_2 + \dots + (n+1){}^nC_n$$

Thus

$$\begin{aligned} 2{}^nC_1 + 3{}^nC_2 + \dots + (n+1){}^nC_n &= n(2)^{n-1} + (2)^n - 1 \\ &= (n+2)2^{n-1} - 1 \end{aligned}$$

(c) $f(x+2) = x^2 + 2$

$$f(x) = (x-2)^2 + 2$$

$$= x^2 - 4x + 4 + 2$$

$$= x^2 - 4x + 6$$

(d)



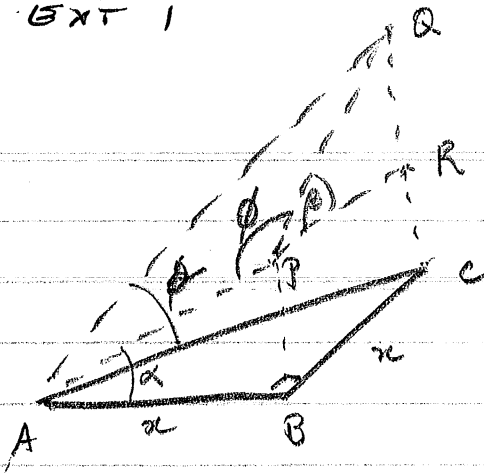
- (i) If J&M sit on the short side, they can be arranged in 12 ways, and the other guests in $7!$ ways. Thus $12 \times 7!$ ways.
If J&M sit on the long side they can be arranged in 20 ways, and the other guests in $7!$ Ways. Thus $20 \times 7!$

Hence there are $32 \times 7! = 161\,280$ ways.

- (ii) If John sits on the short side he has four seats available, and Mary (on the long side) has 5, thus $20 \times 7!$

But Mary may be the one on the short side.

Thus the total is $40 \times 7! = 201\,600$



$$(i) \quad \tan \alpha = \frac{BP}{x} = \frac{BP}{AB}$$

$$\therefore BP = x \tan \alpha$$

$$\tan \beta = \frac{QR}{x} = \frac{QR}{AC}$$

$$\therefore QR = x \tan \beta$$

$$QC = x \tan \alpha + x \tan \beta$$

$$= x (\tan \alpha + \tan \beta)$$

$$\tan \alpha = \frac{QC}{AC}$$

$$= \frac{x (\tan \alpha + \tan \beta)}{\sqrt{2} x}$$

$$= \frac{\tan \alpha + \tan \beta}{\sqrt{2}}$$

GIVEN

(2)

$$(ii) \quad \cos \phi = \frac{AP^2 + PQ^2 - AQ^2}{2 AP \cdot PQ}$$

$$= \frac{\frac{x^2}{\cos^2 \alpha} + \frac{x^2}{\cos^2 \beta} - \frac{2x^2}{\cos^2 \alpha \cos \beta}}{2 \cdot \frac{x}{\cos \alpha} \cdot \frac{x}{\cos \beta}}$$

$$= \frac{\sec^2 \alpha + \sec^2 \beta - 2 (1 + \frac{(\tan \alpha + \tan \beta)^2}{2})}{2 \sec \alpha \sec \beta}$$

$$= \frac{\sec^2 \alpha + \sec^2 \beta - (2 + \tan^2 \alpha + 2 \tan \alpha \tan \beta + \tan^2 \beta)}{2 \sec \alpha \sec \beta}$$

$$= \frac{\sec^2 \alpha + \sec^2 \beta - \sec^2 \alpha - 2 \tan \alpha \tan \beta - \sec^2 \beta}{2 \sec \alpha \sec \beta}$$

$$= -\sin \alpha \sin \beta$$

GIVEN

(2)

$$\ddot{x} = 0$$

$$\dot{x} = 12 \cos 60^\circ$$

$$= 6$$

$$x = 6t$$

$$\ddot{y} = -10$$

$$\dot{y} = -10t + 12 \sin 60^\circ$$

$$= -10t + 6\sqrt{3}$$

$$y = -5t^2 + 6\sqrt{3}t$$

$$\therefore y = -5 \left(\frac{x}{6} \right)^2 + 6\sqrt{3} \times \frac{x}{6}$$

$$= -\frac{5x^2}{36} + \sqrt{3}x$$

$$\text{If } x = -y, -x = -\frac{5x^2}{36} + \sqrt{3}x$$

$$\therefore \frac{5x^2}{36} - (\sqrt{3}+1)x = 0$$

$$\therefore x(5x - 36(\sqrt{3}+1)) = 0$$

$$\therefore x = 0 \quad \text{or} \quad x = \frac{36(\sqrt{3}+1)}{5}$$

$$\therefore x = \frac{36(\sqrt{3}+1)}{5}$$

$$\dot{x} = 6$$

$$\dot{y} = -10 \times \frac{36(\sqrt{3}+1)}{5} + 6\sqrt{3}$$

$$= -12(\sqrt{3}+1) + 6\sqrt{3} = -6\sqrt{3} - 12$$

$$\text{Speed} = \sqrt{36 + (-6\sqrt{3} - 12)^2}$$

$$= \sqrt{36 + 108 + 144 + 144\sqrt{3}}$$

$$= \sqrt{288 + 144\sqrt{3}}$$

$$= 12\sqrt{2 + \sqrt{3}}$$

$$= 23.2$$