2006 *Catholic Schools*

Catholic Schools Trial Examination

Mathematics Extension 1

HSC SUPPORT

06 CT 1a Evaluate $\int_0^{\frac{\pi}{6}} \sec 2x \tan 2x \; dx$. **2**

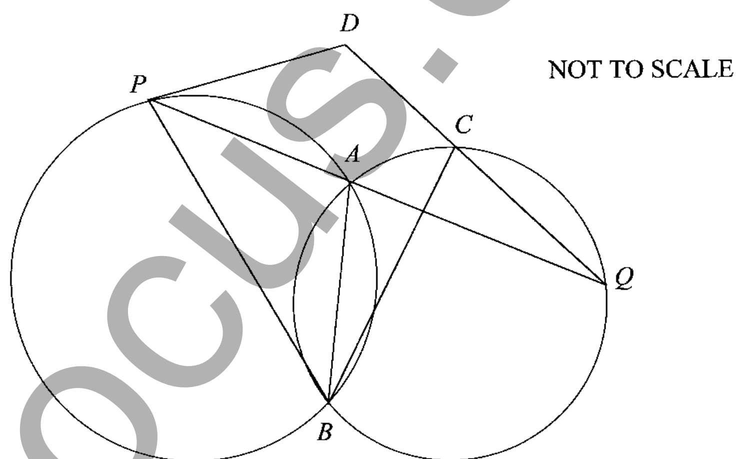
06 CT 1b Find the acute angle between the lines $3x - y - 2 = 0$ and $x + 2y - 3 = 0$. Give the answer correct to the nearest degree. **2**

06 CT 1c The polynomial P(x) is given by $P(x) = x^3 + (k - 1)x^2 + (1 - k)x - 1$ for some real number k.

(i) Show that $x = 1$ is a root of the equation $P(x) = 0$. **1**

(ii) Given that $P(x) = (x - 1)(x^2 + kx + 1)$, find the set of values of k such that the equation $P(x) = 0$ has 3 real roots. **3**

06 CT 1d Two circles intersect at A and B. P is a point on the first circle and Q is a point on the second circle such that PAQ is a straight line. C is a point on the second circle. The line QC produced and the tangent to the first circle at P meet at D.



(i) Copy the diagram.

(ii) Give a reason why $\angle DPA = \angle PBA$. **1**

(iii) Give a reason why $\angle CQA = \angle CBA$. **1**

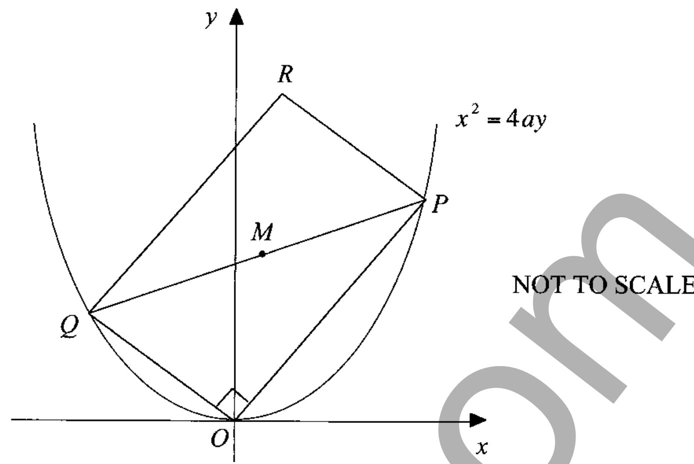
(iv) Hence show that BCDP is a cyclic quadrilateral. **2**

06 CT 2a Show that $\frac{d}{dx} 3^x = 3^x \ln 3$. **2**

06 CT 2b $A(-3, 7)$ and $B(4, -2)$ are two points. Find the coordinates of the point P which divides the interval AB internally in the ratio 3:2. **2**

06 CT 2c Solve the equation $1 + \cos 2x = \sin 2x$ for $0 \le x \le 2\pi$. **4**

06 CT **2d** $P(2ap, ap^2)$ and $Q(2aq, aq^2)$ are two points which move on the parabola $x^2 = 4ay$ such that $\angle POQ = 90^\circ$, where $O(0, 0)$ is the origin. $M\left(a(p+q), \frac{1}{2}a(p^2+q^2)\right)$ is the midpoint of PQ. R is the point such that $OPRQ$ is a rectangle.



(i) Show that $pq = -4$. **1**

(ii) Show that R has coordinates $(a(p+q), a(p^2+q^2))$ **1**

(ii) Find the equation of the locus of R. **2**

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06 CT **3a** Consider the function $f(x) = \dfrac{x^2}{x^2-1}$.

(i) Show that $f(x)$ is an even function. **1**

(ii) Show that $\lim_{x \to \infty} f(x) = 1$. **1**

(iii) Show that the graph $y = f(x)$ has a maximum turning point at the origin. **2**

(iv) Sketch the graph $y = f(x)$ showing clearly the equations of any asymptotes. **2**

(v) The function $g(x)$ is defined by $g(x) = \dfrac{x^2}{x^2-1}$, $x \geq 0$. **2**

Find the equation of the inverse function $g^{-1}(x)$ and state its domain.

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06 CT **3b** Use Mathematical induction to show that for all positive integers $n \geq 1$, **4**

$$\frac{3}{1\times 2\times 2} + \frac{4}{2\times 3\times 2^2} + \frac{n+2}{n(n+1)2^n} = 1 - \frac{1}{(n+1)2^n}.$$

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06 CT **4a** The region in the first quadrant bounded by the curve $y = 2\tan^{-1}x$ and the y axis between $y = 0$ and $y = \frac{\pi}{2}$ is rotated through one complete revolution about the y axis. Find the exact volume of the solid of revolution so formed. **4**

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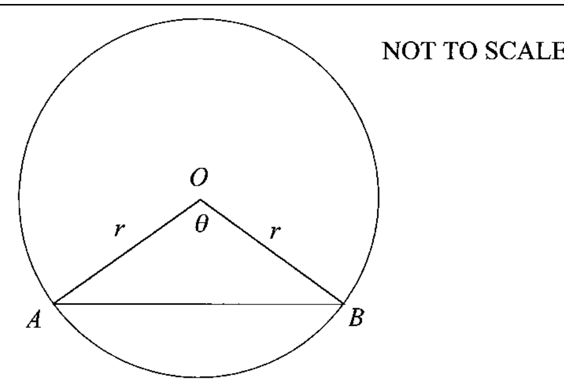
06 CT **4b** AB is a chord of a circle of radius r which subtends an angle θ, $0 < \theta < \pi$, at the centre O. The area of the minor segment cut off by chord AB is one half of the area of the sector AOB.



(i) Show that $\theta - 2\sin\theta = 0$. **2**

(ii) Use an initial approximation $\theta_1 = 2$, and one application of Newton's method **2** to find a second approximation to the value of θ.

Round your answer to 2 decimal places.

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06 CT **4c** Don guesses at random the answers to each of 6 multiple choice questions. In each question there are 3 alternative answers, only one of which is correct. **4**

(i) Find the probability in simplest exact form that Don answers exactly 2 of the 6 questions correctly.

(ii) Find the probability in simplest exact form that the 6^{th} question that Don attempts is only the 2^{nd} question that he answers correctly.

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06 CT **5a** Use the substitution $u = x - 1$ to evaluate $\displaystyle\int_{0.5}^{1.5} \frac{1}{\sqrt{2x - x^2}}\,dx$. **4**

Give the answer in simplest exact form.

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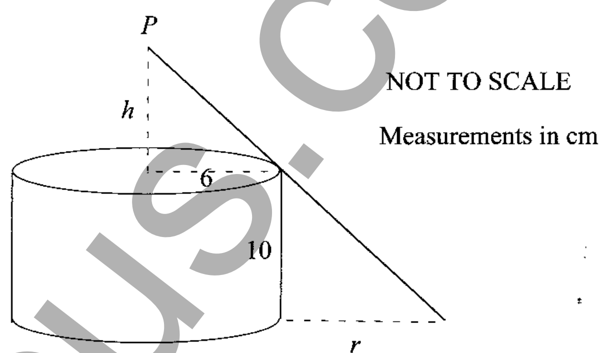
06 CT **5b** A solid wooden cylinder of height 10 cm and radius 6 cm rests with its base on a horizontal table.

A light source P is being lowered vertically downwards from a point above the centre of the top of the cylinder at a constant rate of 0.1 cm s^{-1}.



When the light source is h cm above the top of the cylinder the shadow cast on the table extends r cm from the side of the cylinder.

(i) Show that $r = \dfrac{60}{h}$. **1**

(ii) Find the rate at which r is changing when $h = 5$. **3**

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06 CT **5c** A particle is performing Simple Harmonic Motion in a straight line. At time t seconds it has displacement x metres from a fixed point O on the line, velocity v ms^{-1} given by $v^2 = 32 + 8x - 4x^2$ and acceleration a ms^{-2}.

(i) Find an expression for a in terms of x. **1**

(ii) Find the centre and amplitude of the motion. **2**

(iii) Find the maximum speed of the particle. **1**

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06 CT **6a** At time t minutes the volume flow rate R kilolitres per minute of water into a tank is given by $R = 4\sin^2 t$, $0 \le t \le \pi$.

(i) Find the maximum rate of flow of water into the tank. **1**

(ii) Find the total amount of water which flows into the tank. **3**

Give the answer correct to the nearest litre.

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06 CT **6b** At time t years the number N of individuals in a population is given by $N = A + Be\text{-}t$ for some real constants A and B. After $\ln 2$ years there are 60 individuals and after $\ln 5$ years there are 36 individuals.

(i) Show that A and B satisfy the equations $2A + B = 120$ and $5A + B = 180$. **3**

Hence find the values of A and B.

(ii) Find the limiting population size. **1**

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06 CT **6c** A particle is moving in a straight line. At time t seconds it has displacement x metres from a fixed point O on the line and velocity v ms^{-1} given by $v = \frac{x(2-x)}{2}$. The particle starts 1 metre to the right of O.

(i) Show that $\frac{2}{x(2-x)} = \frac{1}{x} + \frac{1}{2-x}$. **3**

(ii) Find an expression for x in terms of t. **1**

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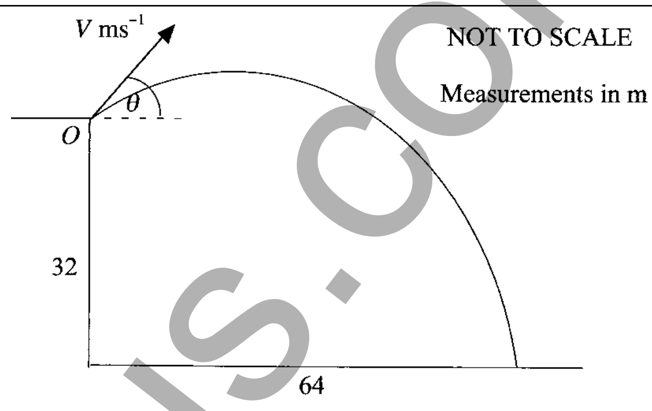
06 CT **7a** A particle is projected with velocity V ms^{-1} at an angle θ above the horizontal from a point O on the edge of a vertical cliff 32 metres above a horizontal beach. The particle moves in a vertical plane under gravity, and 4 seconds later it hits the beach at a point 64 metres from the foot of the cliff.



The acceleration due to gravity is 10ms^{-2}.

(i) Use integration to show that after t seconds the horizontal displacement x metres and the vertical displacement y metres of the particle from O are given by $x = (V\cos\theta)t$ and $y = (V\sin\theta)t - 5t^2$ respectively. **2**

(ii) Write down two equations in V and θ then solve these equations to find the exact value of V and the value of θ in degrees correct to the nearest minute. **3**

(iii) Find the speed of impact with the beach correct to the nearest whole number and the angle of impact with the beach correct to the nearest minute. **3**

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06 CT **7b** (i) Write down the expansion of $x(1 + x)^n$ in ascending powers of x. **1**

(ii) Hence show that $2\,{}^nC_1 + 3\,{}^nC_2 + \ldots + n\,{}^nC_{n-1} = (n + 2)(2^{n-1} - 1)$. **3**

A

1a. $\frac{1}{2}$ **b.** 82^o **c.(ii)** $k \le -2$ or $k \ge 2$ **d.(ii)** ∠ in alt segment **(iii)** ∠ in same segment

2b. $(\frac{6}{5}, \frac{8}{5})$ **2c.** $\frac{\pi}{4}, \frac{\pi}{2}, \frac{5\pi}{4}, \frac{3\pi}{2}$ **2d.** $x^2 = 4a(y - 8a)$ **3a.(v)** $x^2 = \frac{y}{y-1}$, $\{x : x \le 0$ or $x > 1\}$

4a. $\frac{\pi}{2}(4 - \pi)$ **4b.(ii)** 1.90 **4c.(i)** $\frac{80}{243}$ **(ii)** $\frac{80}{729}$ **5a.** $\frac{\pi}{3}$ **5b.(ii)** 0.24 cm s^{-1} **5c.(i)** $a = 4 - 4x$ **(ii)** centre at 1m right of O and amp is 3m **(iii)** 6 ms^{-1} **6a.(i)** 4 kL/min **(ii)** 6283L **6b.(ii)** 20 **6c.(ii)** $x = \frac{2}{1+e^{-t}}$ **7a.(ii)** $V\cos\theta = 16$ and $V\sin\theta = 12$, $V = 20$ and $\theta = 36^o52'$ **(iii)** 32ms^{-1} and $60^o15'$ **7b.(i)** $x + {}^nC_1 x^2 + {}^nC_2 x^3 + \ldots + {}^nC_{n-1} x^n + x^{n+1}$.